Strongly r-matrix induced tensors, Koszul cohomology, and arbitrary-dimensional quadratic Poisson cohomology

Mourad Ammar *, Guy Kass *, Mohsen Masmoudi †, Norbert Poncin *

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Abstract

We introduce the concept of strongly r-matrix induced (SRMI) Poisson structure, report on the relation of this property with the stabilizer dimension of the considered quadratic Poisson tensor, and classify the Poisson structures of the Dufour-Haraki classification (DHC) according to their membership of the family of SRMI tensors. One of the main results of our work is a generic cohomological procedure for SRMI Poisson structures in arbitrary dimension. This approach allows decomposing Poisson cohomology into, basically, a Koszul cohomology and a relative cohomology. Moreover, we investigate this associated Koszul cohomology, highlight its tight connections with Spectral Theory, and reduce the computation of this main building block of Poisson cohomology to a problem of linear algebra. We apply these upshots to two structures of the DHC and provide an exhaustive description of their cohomology. We thus complete our list of data obtained in previous works, see [MP06] and [AP07], and gain fairly good insight into the structure of Poisson cohomology.

Key-words: r-matrix, quadratic Poisson structure, Lichnerowicz-Poisson cohomology, Koszul cohomology, relative cohomology, Spectral Theory

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1 Introduction

In a graded Lie algebra (gLa) $(\mathcal{L}, [.,.])$, $\mathcal{L} = \oplus_i \mathcal{L}^i$, any element with degree 1 that squares to 0, generates a differential graded Lie algebra (dgLa) $(\mathcal{L}, [.,.], \partial_\Lambda)$, $\partial_\Lambda := [\Lambda, .]$, and a gLa $H(\mathcal{L}, [.,.], \partial_\Lambda)$ in cohomology. It is interesting to note that, depending on the initial algebra, such a 2-nilpotent degree 1 element is e.g. an associative algebra structure, a Lie algebra structure, or a Poisson structure, and the associated cohomology is the adjoint Hochschild, the adjoint Chevalley-Eilenberg, and the Lichnerowicz-Poisson (LP) (or simply Poisson) cohomology, respectively. Let us recall that the LP-dgLa is implemented by the shifted Grassmann algebra $(\mathcal{X}(M)[1], \wedge, [.,.]_{\mathrm{SN}})$, $\mathcal{X}(M) = \Gamma(\wedge TM)$, of polyvectors of a manifold M, endowed with the Schouten-Nijenhuis bracket $[.,.]_{\mathrm{SN}}$ (whereas the Hochschild (resp. the Chevalley-Eilenberg) dgLa is generated by the space of multilinear (resp. skew-symmetric multilinear) mappings of the underlying vector space, endowed with the Gerstenhaber (resp. the Nijenhuis-Richardson) graded Lie bracket).

*University of Luxembourg, Campus Limpertsberg, Institute of Mathematics, 162A, avenue de la Faïencerie, L-1511 Luxembourg City, Grand-Duchy of Luxembourg, E-mail: mourad.ammar@uni.lu, guy.kass@uni.lu, norbert.poncin@uni.lu. The research of M. Ammar and N. Poncin was supported by grant R1F105L10. The last author also thanks the Erwin Schrödinger Institute in Vienna for hospitality and support during his visits in 2006 and 2007.

†Université Henri Poincaré, Institut Elie Cartan, B.P. 239, F-54 506 Vandoeuvre-les-Nancy Cedex, France, E-mail: Mohsen.Masmoudi@iecn.u-nancy.fr

Alternatively, LP-cohomology can be viewed as the Lie algebroid (Lad) cohomology of the Lie algebroid $(T^*M, \{.,.\}, \sharp)$ canonically associated with an arbitrary Poisson manifold (M, Λ) (usual notations). The cohomology of a Lad $(E \to M, [\![.,.]\!], \rho)$ (or, equivalently, a Q-structure on a supermanifold) is defined as the cohomology of the Chevalley-Eilenberg subcomplex of the representation $\rho : \Gamma(E) \to \mathrm{Der}(C^\infty(M))$, made up by tensorial cochains. Algebraically, LP-cohomology is defined as the adjoint Chevalley-Eilenberg cohomology of any Poisson-Lie algebra, restricted to the cochain subspace of skew-symmetric multiderivations.

More detailed descriptions of Poisson cohomology can be found e.g. in [Lic77] or [Vai94].

Many papers on Poisson cohomology and Poisson homology, [Kos85], [Bry88], have been published during the last decades. Cohomology of regular Poisson manifolds, [Vai90], [Xu92], (co)homology and resolutions, [Hue90], duality, [Hue97], [Xu97], [ELW99], cohomology in low dimensions or specific cases, [Nak97], [Gin99], [Gam02], [Mon02,1], [Mon02,2], [RV02], [Roy02], [Pic05], extensions of Poisson cohomology, e.g. Lie algebroid cohomology, Jacobi cohomology, Nambu-Poisson cohomology, double Poisson cohomology, and graded Jacobi cohomology, [LMP97], [ILLMP01], [Mon01], [GM03], [LLMP03], [Nak06], [PW07], are only some of the investigated problems. Let us also mention our own works, [MP06], [AP07], in which we suggest an approach to the cohomology of the Poisson tensors of the Dufour-Haraki classification (DHC).

In this paper, we focus on the formal LP-cohomology associated with the quadratic Poisson tensors (QPT) Λ of $\mathbb{R}^n$ that read as real linear combination

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_i \wedge Y_j =: \sum_{i<j} \alpha^{ij} Y_{ij}, \quad \alpha^{ij} \in \mathbb{R} \tag{1}$$

of the wedge products of n commuting linear vector fields $Y_1, \ldots, Y_n$, such that $Y_1 \wedge \ldots \wedge Y_n =: Y_{1\ldots n} \neq 0$. Let us recall that "formal" means that we substitute the space $\mathbb{R}[[x_1, \ldots, x_n]] \otimes \wedge \mathbb{R}^n$ of multivectors with coefficients in the formal series for the usual Poisson cochain space $\mathcal{X}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes \wedge \mathbb{R}^n$. Furthermore, the reader may think about QPT of type (1) as QPT implemented by a classical r-matrix in their stabilizer for the canonical matrix action.

Hence, in Section 2, we are interested in the characterization of the QPT that are images of a classical r-matrix. We comment on the tight relation between the fact that a QPT is induced by an r-matrix and the dimension of its stabilizer. We prove that if the stabilizer of a given QPT Λ of $\mathbb{R}^n$ contains n commuting linear vector fields Y_i, such that $Y_{1\ldots n} \neq 0$, then Λ is implemented by an r-matrix in its stabilizer, see Theorem 1. In the following, we refer to such tensors as strongly r-matrix induced (SRMI) structures and show that any structure of the DHC decomposes into the sum of a major SRMI structure and a small compatible (mostly exact) Poisson tensor, see Theorem 2. This decomposition constitutes the foundation of our cohomological techniques proposed in [MP06] and [AP07]. The preceding description and the philosophy of the mentioned cohomological modus operandi allow understanding that our splitting is in some sense in opposition to the one proven in [LX92] that incorporates the largest possible part of the Poisson tensor into the exact term.

In [MP06], two of us developed a cohomological method in the Euclidean Three-Space that led to a significant simplification of LP-cohomology computations for the SRMI structures of the DHC. Section 3 of the present note aims for extension of this procedure to arbitrary dimensional vector spaces. Nontrivial lemmata allow injecting the space $\mathcal{R}$ of "real" LP-cochains (formal multivector fields) into a larger space $\mathcal{P}$ of "potential" cochains, see Theorem 3, and identifying the natural extension to $\mathcal{P}$ of the LP-differential as the Koszul differential associated with n commuting endomorphisms $X_i - (\operatorname{div} X_i)\,\mathrm{id}$, $X_i = \sum_j \alpha^{ij} Y_j$, $\alpha^{ji} = -\alpha^{ij}$, of the space made up by the polynomials on $\mathbb{R}^n$ with some fixed homogeneous degree, Theorems 4 and 5. We then choose a space $\mathcal{S}$ supplementary to $\mathcal{R}$ in $\mathcal{P}$ and show that the LP-differential induces a differential on $\mathcal{S}$. Eventually, we end up with a short exact sequence of differential spaces and an exact triangle in cohomology. It could be proven that the LP-cohomology ($\mathcal{R}$-cohomology) reduces, essentially, to the above-depicted Koszul cohomology ($\mathcal{P}$-cohomology) and a relative cohomology ($\mathcal{S}$-cohomology), see Theorem 6.

In order to take advantage of these upshots, we investigate in Section 4 the Koszul cohomology associated to n commuting linear operators on a finite-dimensional complex vector space. We prove a

homotopy-type formula, see Proposition 8, and—using spectral properties—we show that the Koszul cohomology is, roughly spoken, located inside (a direct sum of intersections of) the kernels of some transformations that can be constructed recursively from the initially considered operators, Proposition 9 and Corollary 3.

In Section 5, we apply this result, gain valuable insight into the structure of the Koszul cohomology implemented by SRMI tensors, and show that in order to compute this central part of Poisson cohomology it basically suffices to solve triangular systems of linear equations.

Section 6 contains a full description of the LP-cohomology spaces of structures Λ_3 and Λ_9 of the DHC.

Eventually, the aforementioned general upshots and our growing list of explicit data allow describing the main LP-cohomological phenomena, see Section 7.

2 Characterization of strongly r-matrix induced Poisson structures

2.1 Stabilizer dimension and r-matrix generation

Poisson structures implemented by an r-matrix are of importance, e.g. in Deformation Quantization, especially in view of Drinfeld's method. In the following, we report on an idea regarding generation of quadratic Poisson tensors by classical r-matrices.

Set $G = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathrm{gl}(n, \mathbb{R})$. The Lie algebra isomorphism between $\mathfrak{g}$ and the algebra $\mathcal{X}_0^1(\mathbb{R}^n)$ of linear vector fields, extends to a Grassmann algebra and a graded Poisson-Lie algebra homomorphism $J : \wedge\mathfrak{g} \to \oplus_k \left(\mathcal{S}^k\mathbb{R}^{n*} \otimes \wedge^k\mathbb{R}^n\right)$. It is known that its restriction

$$J^k : \wedge^k\mathfrak{g} \to \mathcal{S}^k\mathbb{R}^{n*} \otimes \wedge^k\mathbb{R}^n$$

is onto, but has a non-trivial kernel if $k, n \geq 2$. In particular,

$$J^3[r,r]_{\mathrm{SN}} = [J^2r, J^2r]_{\mathrm{SN}},\ r \in \mathfrak{g}\wedge\mathfrak{g},$$

where $[.,.]_{\mathrm{SN}}$ is the Schouten-Nijenhuis bracket. These observations allow to understand that the characterization of the quadratic Poisson structures that are implemented by a classical r-matrix, i.e. a bimatrix $r \in \mathfrak{g}\wedge\mathfrak{g}$ that verifies the Classical Yang-Baxter Equation $[r,r]_{\mathrm{SN}} = 0$, is an open problem.

Quadratic Poisson tensors Λ_1 and Λ_2 are equivalent if and only if there is $A \in G$ such that $A_*\Lambda_1 = \Lambda_2$, where $*$ denotes the standard action of G on tensors of $\mathbb{R}^n$. As J^2 is a G-module homomorphism, i.e.

$$A_*(J^2r) = J^2(\mathrm{Ad}(A)r),\ A \in G, r \in \mathfrak{g}\wedge\mathfrak{g},$$

the G-orbit of a quadratic Poisson structure $\Lambda = J^2r$ is the pointwise J^2-image of the G-orbit of r. Furthermore, representation Ad acts by graded Lie algebra homomorphisms, i.e.

$$\mathrm{Ad}(A)[r,r]_{\mathrm{SN}} = [\mathrm{Ad}(A)r, \mathrm{Ad}(A)r]_{\mathrm{SN}}.$$

Hence, if $\Lambda = J^2r$, where r is a classical r-matrix, the whole orbit of this quadratic Poisson tensor is made up by r-matrix induced structures.

Of course, any quadratic Poisson tensor Λ is implemented by bimatrices $r \in \mathfrak{g}\wedge\mathfrak{g}$. In order to determine wether the G-orbit O_Λ of this tensor is generated by r-matrices, we have to take an interest in the preimage

$$(J^2)^{-1}(O_\Lambda) = \cup_{r\in(J^2)^{-1}\Lambda} O_r,$$

composed of the G-orbits O_r of all the bimatrices r that are mapped on Λ by J^2. We claim that the chances that a fiber of this bundle is located inside r-matrices are the bigger, the smaller is O_Λ. In other words, the dimension of the isotropy Lie group G_Λ of Λ, or of its Lie algebra, the stabilizer

$$\mathfrak{g}_\Lambda = \{a \in \mathfrak{g} : [\Lambda, Ja]_{\mathrm{SN}} = 0\}$$

of Λ for the corresponding infinitesimal action, should be big enough. In addition to the ostensible intuitive clearness of this conjecture, positive evidence comes from the fact that, in $\mathbb{R}^3$, the Poisson tensor $\Lambda = (x_1^2 + x_2x_3)\partial_{23}$, $\partial_{23} := \partial_2 \wedge \partial_3$, $\partial_i := \partial/\partial_{x_i}$, is not r-matrix induced, see [MMR02], and the dimension of its stabilizer is $\dim \mathfrak{g}_\Lambda = 2$, as well as from the following theorem (we implicitly identify stabilizer $\mathfrak{g}_\Lambda \subset \mathfrak{g}$ and the (isomorphic) Lie subalgebra $J^1\mathfrak{g}_\Lambda = \{Y \in \mathcal{X}_0^1(\mathbb{R}^n) : [\Lambda, Y]_{\mathrm{SN}} = 0\} \subset \mathcal{X}_0^1(\mathbb{R}^n)$ of linear vector fields of $\mathbb{R}^n$).

Theorem 1. *Let Λ be a quadratic Poisson tensor of $\mathbb{R}^n$. If its stabilizer $\mathfrak{g}_\Lambda$ contains n commuting linear vector fields Y_i, $i \in \{1, \ldots, n\}$, such that $Y_1 \wedge \ldots \wedge Y_n \neq 0$, then Λ is implemented by a classical r-matrix that belongs to the stabilizer, i.e. $\Lambda = J^2 a$, $[a, a]_{\mathrm{SN}} = 0$, $a \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$.*

Proof. Let $(x_1, \ldots, x_n)$ be the canonical coordinates of $\mathbb{R}^n$. Set $\partial_r = \partial_{x_r}$ and $Y_i = \sum_{r=1}^n \ell_{ir}\partial_r$, with $\ell \in \mathrm{gl}(\mathrm{n}, \mathbb{R}^{\mathrm{n}*})$. The determinant $D = \det \ell$ does not vanish everywhere, since $Y_{1\ldots n} = D\,\partial_{1\ldots n}$ and $Y_{1\ldots n} \neq 0$. At any point of the nonempty open subset $Z = \{x \in \mathbb{R}^n, D(x) \neq 0\}$ of $\mathbb{R}^n$, the Y_i form a basis of the corresponding tangent space of $\mathbb{R}^n$. Moreover, in Z, we get

$$\partial_{ij} = D^{-2} \sum_{k<l} \left(\mathbf{L}^k_{\ i}\mathbf{L}^l_{\ j} - \mathbf{L}^l_{\ i}\mathbf{L}^k_{\ j}\right) Y_{kl} =: D^{-2} \sum_{k<l} Q^{kl}_{ij} Y_{kl},$$

where $\mathbf{L}$ denotes the matrix of maximal algebraic minors of ℓ, and where $Q^{kl}_{ij} \in \mathcal{S}^{2n-2}\mathbb{R}^{n*}$. Hence, if the quadratic Poisson tensor Λ reads $\Lambda = \sum_{i<j} \Lambda^{ij}\partial_{ij}$, $\Lambda^{ij} \in \mathcal{S}^2\mathbb{R}^{n*}$, we have in Z,

$$\Lambda = D^{-2} \sum_{k<l}\sum_{i<j} \Lambda^{ij} Q^{kl}_{ij} Y_{kl} =: D^{-2} \sum_{k<l} P^{kl} Y_{kl},$$

where $P^{kl} \in \mathcal{S}^{2n}\mathbb{R}^{n*}$. We now prove that the rational functions $D^{-2}P^{kl}$ are actually constants. Since the Y_i are commuting vector fields in $\mathfrak{g}_\Lambda$, the commutation relations $[Y_i, Y_j] = 0$ and $[\Lambda, Y_i]_{\mathrm{SN}} = 0$, $i, j \in \{1, \ldots, n\}$, hold true. It follows that

$$Y_i\left(D^{-2}P^{kl}\right) = \sum_{r=1}^n \ell_{ir}\partial_r\left(D^{-2}P^{kl}\right) = 0, \quad i \in \{1, \ldots, n\},$$

everywhere in Z, and, as ℓ is invertible in Z, that $\partial_r\left(D^{-2}P^{kl}\right) = 0$, $r \in \{1, \ldots, n\}$. Hence, $P^{kl} = \alpha^{kl}D^2$, $\alpha^{kl} \in \mathbb{R}$, in each connected component of Z. As these components are open subsets of $\mathbb{R}^n$, the last result holds in $\mathbb{R}^n$ (in particular the constants α^{kl} associated with different connected components coincide). Eventually,

$$\Lambda = \sum_{k<l} \alpha^{kl} Y_{kl} = J^2\left(\sum_{k<l} \alpha^{kl} a_{kl}\right),$$

where $a_i = (J^1)^{-1}Y_i \in \mathfrak{g}_\Lambda$. Since the a_i are (just as the Y_i) mutually commuting, it is clear that the bimatrix $r = \sum_{k<l} \alpha^{kl} a_{kl} \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$ verifies the classical Yang-Baxter equation. ∎

Definition 1. *We refer to a quadratic Poisson structure Λ that is implemented by a classical r-matrix $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, where $\mathfrak{g}_\Lambda$ denotes the stabilizer of Λ for the canonical matrix action, as a strongly r-matrix induced (SRMI) tensor.*

2.2 Classification theorem in Euclidean Three-Space

Two concepts of exact Poisson structure—tightly related with two special cohomology classes—are used below. Let Λ be a Poisson tensor on a smooth manifold M oriented by a volume element Ω. We say that Λ, which is of course a LP-2-cocycle, is LP-exact (Lichnerowicz-Poisson), if

$$\Lambda = [\Lambda, X]_{\mathrm{SN}},\ X \in \mathcal{X}^1(M),$$

[vector field X is called Liouville vector field and the cohomology class of Λ is the obstruction to infinitesimal rescaling of Λ], and we term Λ K-exact (Koszul), if

$$\Lambda = \delta(T),\ T \in \mathcal{X}^3(M).$$

Operator $\delta := \phi^{-1} \circ d \circ \phi$ is the pullback of the de Rham differential d by the canonical vector space isomorphism $\phi := i.\Omega$. Although introduced earlier, the generalized divergence δ ($\delta(X) = \mathrm{div}_\Omega X,\ X \in \mathcal{X}^1(M)$) is prevalently attributed to J.-L. Koszul. The curl vector field $K(\Lambda) := \delta(\Lambda)$ of Λ (if Ω is the standard volume of $\mathbb{R}^3$ and Λ is identified with a vector field $\vec{\Lambda}$ of $\mathbb{R}^3$, $K(\Lambda)$ coincides with the standard curl $\vec{\nabla} \wedge \vec{\Lambda}$) is an LP-1-cocycle (which maps a function to the divergence of its Hamiltonian vector field, and the cohomology class of which is the well-known modular class of Λ [this class is independent of Ω and is the obstruction to existence on M of a measure preserved by all Poisson automorphisms] that is relevant e.g. in the classification of Poisson structures, see [DH91], [GMP93], [LX92], and in Poincaré duality, see [ELW99], [ILLMP01]). In $\mathbb{R}^n$, $n \geq 3$, a Poisson tensor Λ is K-exact, if and only if it is "irrotational", i.e. $K(\Lambda) = 0$, and in $\mathbb{R}^3$, K-exact means "function-induced", i.e.

$$\Lambda = \Pi_f := \partial_1 f\, \partial_{23} + \partial_2 f\, \partial_{31} + \partial_3 f\, \partial_{12},\ f \in C^\infty(\mathbb{R}^3).$$

The K-exact quadratic Poisson tensors Π_p of $\mathbb{R}^3$, i.e. the K-exact Poisson structures that are induced by a homogeneous polynomial $p \in \mathcal{S}^3\mathbb{R}^{3*}$, represent class 14 of the DHC. The cohomology of this class has been studied in [Pic05] (actually the author deals with structures Π_p implemented by a weight homogeneous polynomial p with an isolated singularity). Hence, class 14 of the DHC will not be examined in the current work.

Let us also recall that two Poisson tensors Λ_1 and Λ_2 are compatible, if their sum is again a Poisson structure, i.e. if $[\Lambda_1, \Lambda_2]_{\mathrm{SN}} = 0$.

The following theorem classifies the quadratic Poisson classes according to their membership of the family of strongly r-matrix induced structures. Furthermore, we show that any structure reads as the sum of a *major* strongly r-matrix induced tensor and a *small* compatible Poisson structure. On one hand, this membership entails accessibility to the cohomological technique exemplified in [MP06], on the other, this splitting—which, by the way, differs from the decomposition suggested in [LX92] in the sense that we incorporate the biggest possible part of the structure into the strongly induced term—is of particular importance with regard to the cohomological approach detailed in [AP07].

Theorem 2. *Let $a, b, c \in \mathbb{R}$ and let Λ_i ($i \in \{1, \ldots, 13\}$) be the quadratic Poisson tensors of the DHC, see [DH91]. Denote the canonical coordinates of $\mathbb{R}^3$ by x, y, z (or x_1, x_2, x_3) and the partial derivatives with respect to these coordinates by $\partial_1, \partial_2, \partial_3$ ($\partial_{ij} = \partial_i \wedge \partial_j$).*

If $\dim \mathfrak{g}_\Lambda > 3$ (subscript i omitted), there are mutually commuting linear vector fields Y_1, Y_2, Y_3, such that

$$\Lambda = \alpha Y_{23} + \beta Y_{31} + \gamma Y_{12} \quad (\alpha, \beta, \gamma \in \mathbb{R}),$$

so that Λ is strongly r-matrix induced (SRMI), i.e. implemented by a classical r-matrix in $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. In the following classification of the quadratic Poisson tensors with regard to property SRMI, we decompose each not SRMI tensor into the sum of a major SRMI structure and a smaller compatible quadratic Poisson tensor.

- *Set $Y_1 = x\partial_1, Y_2 = y\partial_2, Y_3 = z\partial_3$*

  1. $\Lambda_1 = a\, yz\partial_{23} + b\, xz\partial_{31} + c\, xy\partial_{12}$ *is SRMI for all values of the parameters a, b, c. More precisely,*

  $$\Lambda_1 = a\, Y_{23} + b\, Y_{31} + c\, Y_{12}$$

  2. $\Lambda_4 = a\, yz\partial_{23} + a\, xz\partial_{31} + (b\, xy + z^2)\partial_{12}$ *is not SRMI if and only if $(a, b) \neq (0, 0)$. We have,*

  $$\Lambda_4 = a(Y_{23} + Y_{31}) + b\, Y_{12} + \frac{1}{3}\Pi_{z^3}$$

- *Set $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2 - y\partial_1, Y_3 = z\partial_3$*

  1. $\Lambda_2 = (2ax - by)z\partial_{23} + (bx + 2ay)z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for any a, b. More precisely,*

  $$\Lambda_2 = 2a\, Y_{23} + b\, Y_{31} + a\, Y_{12}$$

2. $\Lambda_7 = ((2a+c)x - by)\, z\partial_{23} + (bx + (2a+c)y)\, z\partial_{31} + a(x^2+y^2)\partial_{12}$ *is SRMI for all* a, b, c. *More precisely,*

$$\Lambda_7 = (2a+c)Y_{23} + b\,Y_{31} + a\,Y_{12}$$

3. $\Lambda_8 = a\,xz\partial_{23} + a\,yz\partial_{31} + \left(\frac{a+b}{2}(x^2+y^2) \pm z^2\right)\partial_{12}$ *is not SRMI if and only if* $(a,b) \neq (0,0)$. *We have,*

$$\Lambda_8 = a\,Y_{23} + \frac{a+b}{2}Y_{12} \pm \frac{1}{3}\Pi_{z^3}$$

- *Set* $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2, Y_3 = z\partial_3$

1. $\Lambda_3 = (2x - a\,y)z\partial_{23} + a\,xz\partial_{31} + x^2\partial_{12}$ *is SRMI for any* a. *More precisely,*

$$\Lambda_3 = 2Y_{23} + a\,Y_{31} + Y_{12}$$

2. $\Lambda_5 = ((2a+1)x + y)\, z\partial_{23} - xz\partial_{31} + a\,x^2\partial_{12}$ $(a \neq -\frac{1}{2})$ *is SRMI for any* a. *More precisely,*

$$\Lambda_5 = (2a+1)Y_{23} - Y_{31} + a\,Y_{12}$$

3. $\Lambda_6 = a\,yz\partial_{23} - a\,xz\partial_{31} - \frac{1}{2}x^2\partial_{12}$ *is SRMI for any* a. *More precisely,*

$$\Lambda_6 = -a\,Y_{31} - \frac{1}{2}Y_{12}$$

- *Set* $Y_1 = \mathcal{E} := x\partial_1 + y\partial_2 + z\partial_3, Y_2 = x\partial_2 + y\partial_3, Y_3 = x\partial_3$

1. $\Lambda_9 = (ax^2 - \frac{1}{3}y^2 + \frac{1}{3}xz)\partial_{23} + \frac{1}{3}xy\partial_{31} - \frac{1}{3}x^2\partial_{12}$ *is SRMI for any* a. *More precisely,*

$$\Lambda_9 = a\,Y_{23} - \frac{1}{3}Y_{12}$$

2. $\Lambda_{10} = \left(a\,y^2 - (4a+1)xz\right)\partial_{23} + (2a+1)xy\partial_{31} - (2a+1)x^2\partial_{12}$ *is not SRMI if and only if* $a \neq -\frac{1}{3}$. *We have,*

$$\Lambda_{10} = -(2a+1)Y_{12} + (3a+1)(y^2 - 2xz)\partial_{23}$$

- *Set* $Y_1 = \mathcal{E}, Y_2 = x\partial_2, Y_3 = (a\,x + (3b+1)z)\,\partial_3$

1. $\Lambda_{11} = \left(a\,x^2 + (2b+1)xz\right)\partial_{23} + (b\,x^2 + c\,z^2)\partial_{12}$ $(a = 0)$ *is not SRMI if and only if* $c \neq 0$. *We have,*

$$\Lambda_{11} = Y_{23} + b\,Y_{12} + \frac{c}{3}\Pi_{z^3}$$

2. $\Lambda_{12} = \left(a\,x^2 + (2b+1)xz\right)\partial_{23} + (b\,x^2 + c\,z^2)\partial_{12}$ $(a = 1)$ *is not SRMI if and only if* $c \neq 0$. *We have,*

$$\Lambda_{12} = Y_{23} + b\,Y_{12} + \frac{c}{3}\Pi_{z^3}$$

3. $\Lambda_{13} = \left(a\,x^2 + (2b+1)xz + z^2\right)\partial_{23} + (b\,x^2 + c\,z^2 + 2xz)\partial_{12}$ *is not SRMI for any* a, b, c. *We have,*

$$\Lambda_{13} = Y_{23} + b\,Y_{12} + \Pi_{\frac{c}{3}z^3 + xz^2}$$

Proof. Let us first mention that the specified basic fields Y_1, Y_2, Y_3 have been read in the stabilizers of the considered Poisson tensors, but that we refrain from publishing the often fairly protracted stabilizer-computations. Indeed, once the vector fields Y_i are known, it is easily checked that, in the SRMI cases, they verify the assumptions of Theorem 1. Thus the corresponding Poisson structures are actually SRMI tensors. In order to ascertain that a quadratic Poisson structure Λ is not SRMI, it suffices to prove that $\Lambda \notin J^2(\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda)$. This will be done thereinafter. All the quoted decompositions can be directly verified. In most instances, the twist is obviously Poisson, so that compatibility follows.

In the case of Λ_{10}, the twist $\Lambda_{10,\mathrm{II}} = (y^2 - 2xz)\partial_{23}$ is a non-K-exact Poisson structure. This is a direct consequence of the result $K(\Lambda_{10,\mathrm{II}}) = \vec{\nabla} \wedge \vec{\Lambda}_{10,\mathrm{II}} = -2x\partial_2 - 2y\partial_3 \neq 0$ and the handy formula

$$[P,Q]_{\mathrm{SN}} = (-1)^p D(P \wedge Q) - D(P) \wedge Q - (-1)^p P \wedge D(Q), \forall P \in \mathcal{X}^p(M), Q \in \mathcal{X}^q(M).$$

The statement regarding the dimension of stabilizer $\mathfrak{g}_\Lambda$ is obvious in view of the following main part of this proof.

Denote by E_{ij} $(i,j \in \{1,2,3\})$ the canonical basis of $\mathrm{gl}(3,\mathbb{R})$.

- For Λ_4, if $(a,b) \neq (0,0)$, stabilizer $\mathfrak{g}_{\Lambda_4}$ and the image $J^2(\mathfrak{g}_{\Lambda_4} \wedge \mathfrak{g}_{\Lambda_4})$ are generated by

$$(\frac{1}{2}E_{11} + E_{22}, \frac{1}{2}E_{11} + E_{33}) \quad \text{and} \quad yz\partial_{23} - \frac{1}{2}xz\partial_{31} - \frac{1}{2}xy\partial_{12},$$

  respectively. Hence, Λ_4 is not SRMI.

- For Λ_8, if $(a,b) \neq (0,0)$, the generators of $\mathfrak{g}_{\Lambda_8}$ and $J^2(\mathfrak{g}_{\Lambda_8} \wedge \mathfrak{g}_{\Lambda_8})$ are

$$(E_{11} + E_{22} + E_{33}, E_{12} - E_{21}) \quad \text{and} \quad -xz\partial_{23} - yz\partial_{31} + (x^2+y^2)\partial_{12}.$$

  So Λ_8 is not SRMI.

- For Λ_{10}, if $a \neq -\frac{1}{3}$, the generators are

$$(E_{11} + E_{22} + E_{33}, E_{12} + E_{23}) \quad \text{and} \quad (y^2 - xz)\partial_{23} - xy\partial_{31} + x^2\partial_{12}.$$

- For Λ_{11}, $c \neq 0$, Λ_{12}, $c \neq 0$, and Λ_{13}, the generators are

$$(E_{11} + E_{22} + E_{33}, E_{12}, E_{32}) \quad \text{and} \quad (-xz\partial_{23} + x^2\partial_{12}, z^2\partial_{23} - xz\partial_{12}). \blacksquare$$

Remarks.

- For $\Lambda = \Lambda_i$, $i \in \{11,12,13\}$, $c \neq 0$ if $i \in \{11,12\}$, the dimension of the stabilizer is $\dim \mathfrak{g}_\Lambda = 3$, whereas $J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda = \{0\}$. Hence, if the dimension of the stabilizer coincides with the dimension of the space, the Poisson structure is not necessarily a SRMI tensor.

- For Λ_{10} e.g., the decomposition proved in [LX92] yields

$$\Lambda_{10} = -\frac{1}{3}Y_{12} \ + \ \Pi_{\frac{c}{3}z^3 + xz^2 + (b+\frac{1}{3})x^2z + \frac{a}{3}x^3}.$$

3 Poisson cohomology of quadratic structures in a finite-dimensional vector space

3.1 Koszul homology and cohomology

Let $\wedge = \wedge_n\langle\vec{\eta}\rangle$ be the Grassmann algebra on $n \in \mathbb{N}_0$ generators $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, i.e. the algebra generated over a field $\mathbb{F}$ of characteristic 0 (in this work $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) by generators $\eta_1, \ldots, \eta_n$ subject to the anticommutation relations $\eta_k\eta_\ell + \eta_\ell\eta_k = 0$, $k,\ell \in \{1,\ldots,n\}$. Set $\wedge = \oplus_{p=0}^n \wedge^p$, with obvious notations, and let $\vec{h} = (h_1, \ldots, h_n)$ be dual generators: $i_{h_k}\eta_\ell = \delta_{k\ell}$. We also need the creation operator $e_{\eta_k} : \wedge \ni \omega \to \eta_k\omega \in \wedge$ and the annihilation operator $i_{h_k} : \wedge \ni \omega \to i_{h_k}\omega \in \wedge$, where the interior product is defined as usual. Eventually, we denote by E a vector space over $\mathbb{F}$ and by $\vec{X} = (X_1, \ldots, X_n)$ an n-tuple of commuting linear operators on E.

Definition 2. *The complex*

$$0 \to E \otimes_{\mathbb{F}} \wedge^n \to E \otimes_{\mathbb{F}} \wedge^{n-1} \to \ldots \to E \otimes_{\mathbb{F}} \wedge^1 \to E \to 0,$$

with differential $\kappa_{\vec{X}} = \sum_{k=1}^n X_k \otimes i_{h_k}$*, is the Koszul chain complex (K_*-complex)* $K_*(\vec{X}, E)$ *associated with* $\vec{X}$ *on* E*. The Koszul homology group is denoted by* $KH_*(\vec{X}, E)$*.*

Definition 3. *The complex*

$$0 \to E \to E \otimes_{\mathbb{F}} \wedge^1 \to \ldots \to E \otimes_{\mathbb{F}} \wedge^{n-1} \to E \otimes_{\mathbb{F}} \wedge^n \to 0,$$

with differential $\mathcal{K}_{\vec{X}} = \sum_{k=1}^n X_k \otimes e_{\eta_k}$, is the Koszul cochain complex (K^*-complex) $K^*(\vec{X}, E)$ associated with $\vec{X}$ on E. We denote by $KH^*(\vec{X}, E)$ the corresponding Koszul cohomology group.

Observe that commutation of the X_k and anticommutation of the i_{h_k} (resp. the e_{η_k}) entail that $\kappa_{\vec{X}}$ (resp. $\mathcal{K}_{\vec{X}}$) actually squares to 0.

Example 1. It is easily checked that, if we choose $\mathbb{F} = \mathbb{R}$, $E = C^\infty(\mathbb{R}^3)$, $\eta_k = dx_k$ (resp. $\eta_k = \partial_k = \partial_{x_k}$ and $h_k = dx_k$), and $X_k = \partial_k$ ($k \in \{1, 2, 3\}$, x_1, x_2, x_3 canonical coordinates of $\mathbb{R}^3$), the K^*-complex (resp. the K_*-complex) is nothing but the de Rham complex $(\Omega(\mathbb{R}^3), d)$ (resp. its dual version $(\mathcal{X}(\mathbb{R}^3), \delta)$, see above). Note that, if we identify the subspaces $\Omega^k(\mathbb{R}^3)$ of homogeneous forms with the corresponding spaces of components E, E^3, E^3, E, this K^*-complex reads

$$0 \to E \overset{\mathcal{K}=\vec{\nabla}(.)}{\to} E^3 \overset{\mathcal{K}=\vec{\nabla}\wedge(.)}{\to} E^3 \overset{\mathcal{K}=\vec{\nabla}\cdot(.)}{\to} E \to 0, \tag{2}$$

with self-explaining notations.

Example 2. For $\mathbb{F} = \mathbb{R}$, $E = \mathcal{S}\mathbb{R}^{3*} = \mathbb{R}[x_1, x_2, x_3]$, $\eta_k = \partial_k$, $X_k = \mathfrak{m}_{P_k}$ ($k \in \{1, 2, 3\}$, $P_k \in E^{d_k}$, $d_k \in \mathbb{N}$, $\mathfrak{m}_{P_k} : E \ni Q \to P_k Q \in E$), the chain spaces of the K_*-complex are the spaces of homogeneous polyvector fields on $\mathbb{R}^3$ with polynomial coefficients, and anew identification with the corresponding spaces E, E^3, E^3, E of components, allows to write this K_*-complex in the form

$$0 \to E \overset{\kappa=(.)\vec{P}}{\to} E^3 \overset{\kappa=(.)\wedge\vec{P}}{\to} E^3 \overset{\kappa=(.)\cdot\vec{P}}{\to} E \to 0, \tag{3}$$

where $\vec{P} = (P_1, P_2, P_3)$.

Remarks.

- Of course, the Koszul cohomology and homology complexes defined in Example 1 are exact, expect that $KH^0(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq KH_3(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq \mathbb{R}$.

- Let us recall that an R–regular sequence on a module M over a commutative unit ring R, is a sequence $(r_1, \ldots, r_d) \in R^d$, such that r_k is not a zero divisor on the quotient $M/\langle r_1, \ldots, r_{k-1}\rangle M$, $k \in \{1, \ldots, d\}$, and $M/\langle r_1, \ldots, r_d\rangle M \neq 0$. In particular, $x_1, \ldots, x_d$ is a (maximal length) regular sequence on the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_d]$ (so that this ring has depth d).

  It is well-known that the K_*-complex described in Example 2 is exact, except for surjectivity of $\kappa = (.) \cdot \vec{P}$, if sequence $\vec{P} = (P_1, P_2, P_3)$ is regular for $\mathbb{R}[x_1, x_2, x_3]$. For instance, if $\vec{P} = \vec{\nabla} p$, where p is a homogeneous polynomial with an isolated singularity at the origin, sequence $\vec{P}$ is regular, see [Pic05].

3.2 Poisson cohomology in dimension 3

Set $E := C^\infty(\mathbb{R}^3)$ and identify—as above—the spaces of homogeneous multivector fields in $\mathbb{R}^3$, with the corresponding component spaces: $\mathcal{X}^0(\mathbb{R}^3) \simeq \mathcal{X}^3(\mathbb{R}^3) \simeq E$ and $\mathcal{X}^1(\mathbb{R}^3) \simeq \mathcal{X}^2(\mathbb{R}^3) \simeq E^3$.

Let $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) \in E^3$ be a Poisson tensor and $f \in E, \vec{X} \in E^3, \vec{B} \in E^3, T \in E$ a 0-, 1-, 2-, and 3-cochain of the LP-complex. The following formulæ for the LP-coboundary operator $\partial_{\vec{\Lambda}}$ can be obtained by straightforward computations:

$$\begin{aligned}
\partial^0_{\vec{\Lambda}} f &= \vec{\nabla} f \wedge \vec{\Lambda}, \\
\partial^1_{\vec{\Lambda}} \vec{X} &= (\vec{\nabla} \cdot \vec{X})\vec{\Lambda} - \vec{\nabla}(\vec{X} \cdot \vec{\Lambda}) + \vec{X} \wedge (\vec{\nabla} \wedge \vec{\Lambda}), \\
\partial^2_{\vec{\Lambda}} \vec{B} &= -(\vec{\nabla} \wedge \vec{B}) \cdot \vec{\Lambda} - \vec{B} \cdot (\vec{\nabla} \wedge \vec{\Lambda}), \\
\partial^3_{\vec{\Lambda}} T &= 0.
\end{aligned}$$

If we denote the differential detailed in Equation (2) (resp. in Equation (3) if $\vec{P}=\vec{\Lambda}$, in Equation (3) if $\vec{P}=\vec{\nabla}\wedge\vec{\Lambda}$) by $\mathcal{K}$ (resp. κ', κ''), we get

$$\partial^0_{\vec{\Lambda}}=\kappa'\mathcal{K},\partial^1_{\vec{\Lambda}}=\kappa'\mathcal{K}-\mathcal{K}\kappa'+\kappa'',\partial^2_{\vec{\Lambda}}=-\kappa'\mathcal{K}-\kappa'',\partial^3_{\vec{\Lambda}}=0. \tag{4}$$

As aforementioned, investigations are confined in this paper to quadratic Poisson tensors and polynomial (or formal) LP-cochains. If structure $\vec{\Lambda}$ is K-exact, i.e., in view of notations due to the elimination of the module basis of multivector fields, $\vec{\Lambda}=\vec{\nabla}p \quad (p\in\mathcal{S}^3\mathbb{R}^{3*})\ \Leftrightarrow\ \vec{\nabla}\wedge\vec{\Lambda}=0$, homology operator κ'' vanishes. If, moreover, p has an isolated singularity (IS), not only the K^*-complex associated with $\mathcal{K}$ is exact up to injectivity of $\mathcal{K}=\vec{\nabla}(.)$, but also the K_*-complex associated with κ' is, see above, acyclic up to surjectivity of $\kappa'=(.)\cdot\vec{\Lambda}$. In [Pic05], the author has computed inter alia the LP-cohomology for a weight-homogeneous polynomial p with an IS.

Below, we describe a generic cohomological technique for SRMI Poisson tensors in a finite-dimensional vector space. This approach extends Formulæ (4) to dimension n and reduces simultaneously the LP-coboundary operator ∂_Λ to a single Koszul differential.

3.3 Poisson cohomology in dimension n

We denote by L the matrix of maximal minors of a matrix $\ell\in\mathrm{gl}(n,\mathbb{R}^{n*})$ (or of a matrix with entries in a field $\mathbb{F}$ of non-zero characteristic), so L_{ij} is the minor of ℓ obtained by cancellation of line i and column j. More generally, if $\nu=\{1,\ldots,n\}$, $\mathbf{i}=(i_1,\ldots,i_m)\in\nu^m$ $(i_1<\ldots<i_m,\ m\in\{1,\ldots,n\})$, we denote by $\mathbf{I}=(I_1,\ldots,I_{n-m})$ the complement of $\mathbf{i}$ in ν. If $\mathbf{j}=(j_1,\ldots,j_m)$ is an m-tuple similar to $\mathbf{i}$, we denote by $L_{\mathbf{ij}}$ the minor of ℓ obtained by cancellation of the lines $\mathbf{i}$ and the columns $\mathbf{j}$, and by $L^{\mathbf{ij}}$ the minor of ℓ at the intersections of lines $\mathbf{i}$ and columns $\mathbf{j}$. Hence, $L_{\mathbf{ij}}=L^{\mathbf{IJ}}$ and $L_{\mathbf{IJ}}=L^{\mathbf{ij}}$. Moreover, $D=\det\ell\in\mathcal{S}^n\mathbb{R}^{n*}$ is the determinant of ℓ, $\mathcal{L}$ stands for the matrix of maximal minors of $L\in\mathrm{gl}(n,\mathcal{S}^{n-1}\mathbb{R}^{n*})$, and we apply the just introduced notations $\mathcal{L}_{\mathbf{ij}}$ and $\mathcal{L}^{\mathbf{ij}}$ also to $\mathcal{L}$. Eventually, as already mentioned above, $\mathbf{L}$ denotes the matrix of algebraic maximal minors of ℓ.

Remark. In the following, we systematically assume that $D\neq 0$, i.e. that polynomial D does not vanish everywhere.

Lemma 1. *For any $m\in\{1,\ldots,n-1\}$ and for any $\mathbf{i}=(i_1,\ldots,i_m)$, $\mathbf{j}=(j_1,\ldots,j_m)$ as above, we have*

$$\mathcal{L}_{\mathbf{ij}}=D^{n-m-1}L^{\mathbf{ij}}\ \text{and}\ \mathcal{L}^{\mathbf{ij}}=D^{m-1}L_{\mathbf{ij}}.$$

The first (resp. second) equation also holds for $m=0$ (resp. $m=n$). In this case it just means that $\det L=D^{n-1}$.

Proof. Of course, the second statement is nothing but a reformulation of the first. We prove the first assertion by induction on m. For $m=n-1$, the assertion is obvious. Indeed, the both sides coincide with the element L_{IJ} of L at the intersection of the line I and the column J. Assume now that the equation holds true for $2\le m\le n-1$ and take any $\mathbf{i}=(i_1,\ldots,i_{m-1})$ and $\mathbf{j}=(j_1,\ldots,j_{m-1})$ of length $m-1$. Let i_m be an (arbitrary) element of $(n-m+1)$-tuple $\mathbf{I}$. We will also have to consider the m-tuple $\underline{\mathbf{i}}=(i_1,\ldots,i_m,\ldots,i_{m-1})$, where the elements have of course been written in the natural order $i_1<\ldots<i_m<\ldots<i_{m-1}$. The rank of i_m inside $\mathbf{I}$ and $\underline{\mathbf{i}}$ will be denoted be $r_{\mathbf{I}}(i_m)$ and $r_{\underline{\mathbf{i}}}(i_m)$ respectively. Using these notations, we get

$$\mathcal{L}_{\mathbf{ij}}=\mathcal{L}^{\mathbf{IJ}}=\sum_{j_m\in\mathbf{J}}(-1)^{r_{\mathbf{I}}(i_m)+r_{\mathbf{J}}(j_m)}L_{i_mj_m}\mathcal{L}_{\underline{\mathbf{ij}}}.$$

Applying the induction assumption, we see that $\mathcal{L}_{\underline{\mathbf{ij}}}=D^{n-m-1}L^{\underline{\mathbf{ij}}}$, so that

$$\mathcal{L}_{\mathbf{ij}}=D^{n-m-1}\sum_{j_m\in\nu}(-1)^{r_{\mathbf{I}}(i_m)+r_{\mathbf{J}}(j_m)}L_{i_mj_m}\sum_{\sigma\in\mathcal{P}(\underline{\mathbf{j}})}\operatorname{sign}\sigma\,\ell_{i_1\sigma_{j_1}}\ldots\ell_{i_m\sigma_{j_m}}\ldots\ell_{i_{m-1}\sigma_{j_{m-1}}},$$

where $\mathcal{P}(\underline{\mathbf{j}})$ is the permutation group of $\underline{\mathbf{j}}$, and where the first sum could be extended to all $j_m\in\nu$, as for $j_m\in\mathbf{j}$ the last determinant vanishes. It is clear that we obtain all the permutations σ of $\underline{\mathbf{j}}$, if

we assign j_m to i_p ($p \in \{1, \ldots, m\}$) and, for each choice of p, all the permutations $\mu \in \mathcal{P}(\mathbf{j})$ to the remaining subscripts i_q. Observe that the signature of the permutation σ that associates j_m with i_p and permutes $\mathbf{j}$ by μ, is $\operatorname{sign}\sigma = (-1)^{r_{\underline{\mathbf{i}}}(i_p) - r_{\underline{\mathbf{j}}}(j_m)} \operatorname{sign}\mu$. Hence, we get

$$\begin{gathered}\mathcal{L}_{\mathbf{ij}} = \\ D^{n-m-1} \sum_{p=1}^{m} \sum_{\mu \in \mathcal{P}(\mathbf{j})} \operatorname{sign}\mu\, \ell_{i_1 \mu_{j_1}} \ldots \widehat{\ell_{i_p j_m}} \ldots \\ \ell_{i_{m-1}\mu_{j_{m-1}}} \sum_{j_m \in \nu} (-1)^{r_{\mathbf{I}}(i_m) + r_{\underline{\mathbf{i}}}(i_p) + r_{\mathbf{J}}(j_m) - r_{\underline{\mathbf{j}}}(j_m)} \ell_{i_p j_m} L_{i_m j_m}.\end{gathered}$$

Remark now that the exponent of -1 can be replaced by

$$r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m) + r_{\underline{\mathbf{i}}}(i_m) + r_{\mathbf{I}}(i_m) + r_{\underline{\mathbf{j}}}(j_m) + r_{\mathbf{J}}(j_m) \sim r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m) + i_m + j_m.$$

Thus, the last sum reads $(-1)^{r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m)} \sum_{j_m \in \nu} (-1)^{i_m + j_m} \ell_{i_p j_m} L_{i_m j_m}$. If $p \neq m$, this sum vanishes, and if $p = m$ it coincides with determinant D. Eventually, we find

$$\mathcal{L}_{\mathbf{ij}} = D^{n-m} \sum_{\mu \in \mathcal{P}(\mathbf{j})} \operatorname{sign}\mu\, \ell_{i_1 \mu_{j_1}} \ldots \widehat{\ell_{i_m j_m}} \ldots \ell_{i_{m-1}\mu_{j_{m-1}}} = D^{n-m} L^{\mathbf{ij}}. \quad \blacksquare$$

Definition 4. *Let $Y_i = \sum_r \ell_{ir} \partial_r$ be n linear vector fields in $\mathbb{R}^n$. Set*

$$\mathcal{R} = \oplus_{p=0}^{n} \mathcal{R}^p = \oplus_{p=0}^{n} \mathbb{R}[[x_1, \ldots, x_n]] \otimes \wedge_n^p \langle \vec{\partial} \rangle$$

and

$$\mathcal{P} = \oplus_{p=0}^{n} \mathcal{P}^p = D^{-1} \oplus_{p=0}^{n} \mathbb{R}[[x_1, \ldots, x_n]] \otimes \wedge_n^p \langle \vec{Y} \rangle,$$

where $D = \det \ell$ and where $\wedge_n^p \langle \vec{\partial} \rangle$ and $\wedge_n^p \langle \vec{Y} \rangle$ are the terms of degree p of the Grassmann algebras on generators $\vec{\partial} = (\partial_1, \ldots, \partial_n)$ and $\vec{Y} = (Y_1, \ldots, Y_n)$ respectively. Space $\mathcal{R}$ (resp. $\mathcal{P}$) is the space of real formal LP-cochains (resp. potential formal LP-cochains).

Remark. The space of polyvector fields $Y_{\mathbf{k}} = Y_{k_1 \ldots k_p} = Y_{k_1} \wedge \ldots \wedge Y_{k_p}$ ($k_1 < \ldots < k_p, p \in \{0, \ldots, n\}$) with coefficients in the quotients by D of formal power series in $(x_1, \ldots, x_n)$, is a concrete model of space $\mathcal{P}$. Indeed, observe first that these spaces are bigraded by the "exterior degree" p and the (total) "polynomial degree", say r. If such a polyvector field vanishes, its homogeneous terms $D^{-1} \sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}}$ ($P^{\mathbf{k}r} \in \mathcal{S}^r \mathbb{R}^{n*}$) vanish. If we decompose the Y_i ($i \in \{1, \ldots, n\}$) in the natural basis ∂_i, we immediately see that the sums $\sum_{\mathbf{k}} L^{\mathbf{ki}} P^{\mathbf{k}r}$ vanish for all $\mathbf{i} = (i_1, \ldots, i_p)$ ($i_1 < \ldots < i_p$). Since these sums can be viewed as the product of a matrix with polynomial entries and the column made up by the $P^{\cdot r}$, the column vanishes outside the vanishing set V of the homogeneous polynomial determinant of this matrix. As the complement of (the conic closed) subset V of $\mathbb{R}^n$ is dense in $\mathbb{R}^n$, the polynomials $P^{\mathbf{k}r}$ vanish everywhere.

Theorem 3. *(i) There is a canonical non surjective injection $i : \mathcal{R} \to \mathcal{P}$ from $\mathcal{R}$ into $\mathcal{P}$.*
(ii) A homogeneous potential cochain $D^{-1} \sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}}$ [of bidegree (p, r)] is real if and only if the [$n!/p!(n-p)!$] homogeneous polynomials $\sum_{\mathbf{k}} L^{\mathbf{ki}} P^{\mathbf{k}r}$ [of degree $p + r$] are divisible by D (for $p = 0$ this condition means that P^r be divisible by D).

Proof. Take a real cochain $C^p = \sum_{\mathbf{i}} \varsigma^{\mathbf{i}} \partial_{\mathbf{i}} \in \mathcal{R}^p$, where, as above, $\mathbf{i} = (i_1, \ldots, i_p)$, $i_1 < \ldots < i_p$. As $\partial_j = D^{-1} \sum_k \mathbf{L}_{kj} Y_k$, we get

$$\partial_{\mathbf{i}} = D^{-p} \sum_{k_1, \ldots, k_p} \mathbf{L}_{k_1 i_1} \ldots \mathbf{L}_{k_p i_p} Y_{k_1 \ldots k_p} = D^{-p} \sum_{k_1 < \ldots < k_p} \left(\sum_{\sigma \in \mathcal{P}(\mathbf{k})} \operatorname{sign}\sigma\, \mathbf{L}_{\sigma_{k_1} i_1} \ldots \mathbf{L}_{\sigma_{k_p} i_p} \right) Y_{k_1 \ldots k_p}.$$

If $|\mathbf{i}| = \sum_{j=1}^{p} i_j$, it follows from Lemma 1, that the determinant in the above bracket is given by

$$(-1)^{|\mathbf{i}| + |\mathbf{k}|} \mathcal{L}^{\mathbf{ki}} = (-1)^{|\mathbf{i}| + |\mathbf{k}|} D^{p-1} L_{\mathbf{ki}},$$

so that

$$C^p = D^{-1} \sum_{\mathbf{k}} \left(\sum_{\mathbf{i}} (-1)^{|\mathbf{i}| + |\mathbf{k}|} L_{\mathbf{ki}} \varsigma^{\mathbf{i}} \right) Y_{\mathbf{k}},$$

where the RHS is in $\mathcal{P}^p$.

Point (ii) is a direct consequence of the preceding remark. ∎

Remark. In view of this theorem, the bigrading $\mathcal{P} = \oplus_{p=0}^n \oplus_{r=0}^\infty \mathcal{P}^{pr}$, defined on $\mathcal{P}$ by the exterior degree and the polynomial degree, induces a bigrading $\mathcal{R} = \oplus_{p=0}^n \oplus_{r=0}^\infty \mathcal{R}^{pr}$ on $\mathcal{R}$.

Consider now a quadratic Poisson tensor Λ in $\mathbb{R}^n$. In the following, we *assume* that Λ is SRMI, and more precisely that there are n mutually commuting linear vector fields $Y_i = \sum_{r=1}^n \ell_{ir}\partial_r$, $\ell \in \mathrm{gl}(n, \mathbb{R}^{n*})$, such that $D = \det \ell \neq 0$ and

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij} \quad (\alpha^{ij} \in \mathbb{R}).$$

Proposition 1. *The determinant $D = \det \ell \in \mathcal{S}^n\mathbb{R}^{n*} \setminus \{0\}$ of ℓ is the unique joint eigenvector of the Y_i with eigenvalues $\operatorname{div} Y_i \in \mathbb{R}$, i.e., D is, up to multiplication by nonzero constants, the unique nonzero polynomial of $\mathbb{R}^n$ that verifies*

$$Y_i D = (\operatorname{div} Y_i) D, \forall i \in \{1, \ldots, n\}.$$

Moreover, if $D = D_1 D_2$, where $D_1 \in \mathcal{S}^{n_1}\mathbb{R}^{n*}$ and $D_2 \in \mathcal{S}^{n_2}\mathbb{R}^{n*}$ $(n_1 + n_2 = n)$ are two polynomials without common divisor, these factors D_1 and D_2 are also joint eigenvectors. If λ_i and μ_i denote their eigenvalues, we have $\lambda_i + \mu_i = \operatorname{div} Y_i$.

Proof. Set $Y_i = \sum_r \ell_{ir}\partial_r = \sum_{rs} a_{ir}^s x_s \partial_r$, $a_{ir}^s \in \mathbb{R}$. Note first that $Y_i(\ell_{jr}) = \sum_t a_{jr}^t \ell_{it}$, and that $[Y_i, Y_j] = 0$ means $Y_i(\ell_{jr}) = Y_j(\ell_{ir})$, for all $i, j, r \in \{1, \ldots, n\}$. If $\mathcal{P}_n$ denotes the permutation group of $\{1, \ldots, n\}$, we then get

$$\begin{aligned} Y_i D &= \sum_{k=1}^n \sum_{\sigma \in \mathcal{P}_n} \operatorname{sign}\sigma \; \ell_{\sigma_1 1} \ldots Y_i(\ell_{\sigma_k k}) \ldots \ell_{\sigma_n n} \\ &= \sum_{k=1}^n \sum_{\sigma \in \mathcal{P}_n} \operatorname{sign}\sigma \; \ell_{\sigma_1 1} \ldots Y_{\sigma_k}(\ell_{ik}) \ldots \ell_{\sigma_n n} \\ &= \sum_{k,t=1}^n a_{ik}^t \sum_{\sigma \in \mathcal{P}_n} \operatorname{sign}\sigma \; \ell_{\sigma_1 1} \ldots \ell_{\sigma_k t} \ldots \ell_{\sigma_n n}. \end{aligned}$$

This last sum vanishes if $k \neq t$ since two columns coincide in this determinant. Eventually, we have

$$Y_i D = \left(\sum_k a_{ik}^k \right) D = (\operatorname{div} Y_i)\, D.$$

As for uniqueness, suppose that there is another polynomial $P \in \mathcal{S}\mathbb{R}^{n*} \setminus \{0\}$, such that $Y_i P = (\operatorname{div} Y_i) P$, for all $i \in \{1, \ldots, n\}$. Then $Y_i\,(P/D) = 0$ in $Z = \{x \in \mathbb{R}^n, D(x) \neq 0\}$ and the same reasoning as in the proof of Theorem 1 allows concluding that there exists $\alpha \in \mathbb{R}^*$ such that $P = \alpha D$.

The assertion concerning the factorization of D is easily understood. Indeed, since $((\operatorname{div} Y_i) D_1 - Y_i D_1) D_2 = D_1 (Y_i D_2)$ and as the polynomials D_1 and D_2 have no common divisor, $Y_i D_2 = P D_2$ and $(\operatorname{div} Y_i) D_1 - Y_i D_1 = Q D_1$, where $P = Q$ is a polynomial. Looking at degrees, we immediately see that $P = Q$ is necessarily constant. ∎

Remark. Observe that the eigenvalues $\operatorname{div} Y_i$, $i \in \{1, \ldots, n\}$, cannot vanish simultaneously. Indeed, in this case, polynomial $D \in \mathcal{S}^n\mathbb{R}^{n*} \setminus \{0\}$, $n \in \mathbb{N}^*$, vanishes everywhere.

Definition 5. *The complex*

$$0 \to \mathcal{R}^0 \to \mathcal{R}^1 \to \ldots \to \mathcal{R}^n \to 0$$

with differential $\partial_\Lambda = [\Lambda, .]_{\mathrm{SN}}$, is the formal LP-complex of Poisson tensor $\Lambda \in \mathcal{S}^2\mathbb{R}^{n*} \otimes \wedge^2\mathbb{R}^n$. We denote the corresponding cohomology groups by $LH^*(\mathcal{R}, \Lambda)$.

The next theorem shows that if the cochains $C \in \mathcal{R}$ are read as $C = iC \in \mathcal{P}$, the LP-differential assumes a simplified shape.

Theorem 4. *Set $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ji} = -\alpha^{ij}$, and $X_i = \sum_{j \neq i} \alpha^{ij} Y_j$.*
(i) Let

$$C = D^{-1} \sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}} \in \mathcal{P}^{pr}$$

be a homogeneous potential cochain. The LP-coboundary of C is given by

$$\partial_\Lambda C = \sum_{\mathbf{k}i} X_i \big(D^{-1} P^{\mathbf{k}r}\big)\, Y_i \wedge Y_{\mathbf{k}} = D^{-1} \sum_{\mathbf{k}i} (X_i - \delta_i \,\mathrm{id})(P^{\mathbf{k}r})\, Y_i \wedge Y_{\mathbf{k}} \in \mathcal{P}^{p+1,r}, \tag{5}$$

where $\delta_i = \mathrm{div}\, X_i \in \mathbb{R}$.

(ii) The LP-coboundary operator ∂_Λ endows $\mathcal{P}$ with a differential complex structure, and preserves the polynomial degree r. This LP-complex of Λ over $\mathcal{P}$ contains the LP-complex $(\mathcal{R}, \partial_\Lambda)$ of Λ over $\mathcal{R}$ as a differential sub-complex.

Proof. Note first that if $C = f\mathbf{Y}$, where f a function and $\mathbf{Y}$ a wedge product of vector fields Y_k, we get

$$\partial_\Lambda(f\mathbf{Y}) = [\Lambda, f\mathbf{Y}]_{\mathrm{SN}} = [\Lambda, f]_{\mathrm{SN}} \wedge \mathbf{Y}, \tag{6}$$

since the Y_k are mutually commuting. However,

$$[\Lambda, f]_{\mathrm{SN}} = \sum_{i<j} \alpha^{ij}\, ((Y_j f) Y_i - (Y_i f) Y_j) = \sum_i (\sum_{j \neq i} \alpha^{ij}\, Y_j f) Y_i = \sum_i (X_i f) Y_i. \tag{7}$$

When combining Equations (6) and (7), we get the first part of Equation (5), whereas its second part is the consequence of Proposition 1. ∎

Corollary 1. *The LP-cohomology groups of Λ over $\mathcal{R}$ and $\mathcal{P}$ are bigraded, i.e.*

$$LH(\mathcal{R}, \Lambda) = \oplus_{r=0}^{\infty} \oplus_{p=0}^{n} LH^{pr}(\mathcal{R}, \Lambda) \quad \textit{and} \quad LH(\mathcal{P}, \Lambda) = \oplus_{r=0}^{\infty} \oplus_{p=0}^{n} LH^{pr}(\mathcal{P}, \Lambda),$$

where for instance $LH^{pr}(\mathcal{P}, \Lambda)$ is defined by

$$LH^{pr}(\mathcal{P}, \Lambda) = \ker(\partial_\Lambda : \mathcal{P}^{pr} \to \mathcal{P}^{p+1,r}) / \operatorname{im}(\partial_\Lambda : \mathcal{P}^{p-1,r} \to \mathcal{P}^{pr}).$$

In the following we deal with the terms $LP^{*r}(\mathcal{P}, \Lambda) = \oplus_{p=0}^{n} LP^{pr}(\mathcal{P}, \Lambda)$ of the LP-cohomology over $\mathcal{P}$ and with the corresponding part of LP-cohomology the subcomplex $\mathcal{R}$.

Theorem 5. *Let E_r be the real finite-dimensional vector space $\mathcal{S}^r \mathbb{R}^{n*}$, and let $\vec{X}_\delta := (X_1 - \delta_1 \,\mathrm{id}, \ldots, X_n - \delta_n \,\mathrm{id})$, $\delta_i = \mathrm{div}\, X_i$, be the n-tuple of the commuting linear operators $X_i - \delta_i \,\mathrm{id}$ on E_r defined in Theorem 4. The LP-cohomology space $LH^{*r}(\mathcal{P}, \Lambda)$ of Λ over $\mathcal{P}$ coincides with the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$ associated with $\vec{X}_\delta$ on E_r:*

$$LH^{*r}(\mathcal{P}, \Lambda) \simeq KH^*(\vec{X}_\delta, E_r).$$

Proof. Direct consequence of result $\partial_\Lambda = \sum_i (X_i - \delta_i \,\mathrm{id}) \otimes e_{Y_i}$ proved in Theorem 4. ∎

In order to study the LP-cohomology group $LH^{\cdot r}(\mathcal{R}, \Lambda)$ of the quadratic Poisson tensor Λ over the formal cochain space $\mathcal{R}$, we introduce a long cohomology exact sequence.

Let $\mathcal{S}^{pr}$ be a complementary vector subspace of $\mathcal{R}^{pr}$ in $\mathcal{P}^{pr}$: $\mathcal{P}^{pr} = \mathcal{R}^{pr} \oplus \mathcal{S}^{pr}$. Space $\mathcal{S} = \oplus_{r=0}^{\infty} \oplus_{p=0}^{n} \mathcal{S}^{pr}$ can easily be promoted into the category of differential spaces. Indeed, denote by $p_{\mathcal{R}}$ and $p_{\mathcal{S}}$ the projections of $\mathcal{P}$ onto $\mathcal{R}$ and $\mathcal{S}$ respectively, and set for any $s \in \mathcal{S}$,

$$\phi s = p_{\mathcal{R}} \partial_\Lambda s, \tilde{\partial}_\Lambda s = p_{\mathcal{S}} \partial_\Lambda s.$$

Proposition 2. *(i) The endomorphism $\tilde{\partial}_\Lambda \in \mathrm{End}_{\mathbb{R}}\mathcal{S}$ is a differential on $\mathcal{S}$, which has weight $(1,0)$ with respect to the bigrading of $\mathcal{S}$, i.e. $\tilde{\partial}_\Lambda : \mathcal{S}^{pr} \to \mathcal{S}^{p+1,r}$.*
(ii) The linear map $\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{S},\mathcal{R})$ is an anti-homomorphism of differential spaces from $(\mathcal{S},\tilde{\partial}_\Lambda)$ into $(\mathcal{R},\partial_\Lambda)$. Its weight with respect to the bidegree is $(1,0)$, i.e. $\phi : \mathcal{S}^{pr} \to \mathcal{R}^{p+1,r}$.
(iii) The sequence $0 \to \mathcal{R} \xrightarrow{i} \mathcal{P} \xrightarrow{p_{\mathcal{S}}} \mathcal{S} \to 0$ is a short exact sequence of homomorphisms of differential spaces, which preserve the bidegree. It induces an exact triangle in cohomology, whose connecting homomorphism $\phi_\sharp$ is canonically implemented by ϕ. If $LH^{pr}(\mathcal{S},\tilde{\Lambda})$ denotes the degree (p,r) term of the cohomology space of the complex $(\mathcal{S},\tilde{\partial}_\Lambda)$, we have $\phi_\sharp : LH^{pr}(\mathcal{S},\tilde{\Lambda}) \to LH^{p+1,r}(\mathcal{R},\Lambda)$.
(iv) The sequence

$$
\begin{aligned}
&0 \to LH^{0r}(\mathcal{R},\Lambda) \xrightarrow{i_\sharp} \dots \\
&\quad \xrightarrow{\phi_\sharp} LH^{pr}(\mathcal{R},\Lambda) \xrightarrow{i_\sharp} LH^{pr}(\mathcal{P},\Lambda) \xrightarrow{(p_{\mathcal{S}})_\sharp} LH^{pr}(\mathcal{S},\tilde{\Lambda}) \xrightarrow{\phi_\sharp} LH^{p+1,r}(\mathcal{R},\Lambda) \xrightarrow{i_\sharp} \dots \xrightarrow{(p_{\mathcal{S}})_\sharp} LH^{nr}(\mathcal{S},\tilde{\Lambda}) \to 0
\end{aligned}
$$

is a long exact cohomology sequence of vector space homomorphisms.
(v) If $\ker^{pr}\phi_\sharp$ and $\mathrm{im}^{p+1,r}\phi_\sharp$ denote the kernel and the image of the restricted map $\phi_\sharp : LH^{pr}(\mathcal{S},\tilde{\Lambda}) \to LH^{p+1,r}(\mathcal{R},\Lambda)$, we have

$$
LH^{pr}(\mathcal{R},\Lambda) \simeq LH^{p-1,r}(\mathcal{S},\tilde{\Lambda})/\ker^{p-1,r}\phi_\sharp \oplus LH^{pr}(\mathcal{P},\Lambda)/\ker^{pr}\phi_\sharp. \tag{8}
$$

Proof. Statements (i) and (ii) are direct consequences of equation $\partial_\Lambda^2 = 0$. For (iii), we only need check that linear map $\phi_\sharp$ coincides with the connecting homomorphism, what is obvious. Eventually, assertion (v) is a corollary of exactness of the long cohomology sequence. ∎

We now identify the $\mathcal{S}$-cohomology with a relative cohomology. Several concepts of relative cohomology can be met in literature. Below, we use the following definition.

Definition 6. *Let V be a vector space endowed with a differential ∂, and let W be a ∂-closed subspace of V. Denote by $\overline{\partial}$ the differential canonically induced by ∂ on the quotient space V/W. The cohomology of the differential space $(V/W,\overline{\partial})$ is called the relative cohomology of (V,W,∂). It is denoted by $H(V,W,\partial)$.*

Proposition 3. *The cohomology induced by ∂_Λ on $\mathcal{S}$ (i.e. the cohomology of differential space $(\mathcal{S},\tilde{\partial}_\Lambda)$) coincides with the relative cohomology of $(\mathcal{P},\mathcal{R},\Lambda)$ (i.e. the cohomology of space $(\mathcal{P}/\mathcal{R},\overline{\partial}_\Lambda)$):*

$$
LH(\mathcal{S},\tilde{\Lambda}) \simeq LH(\mathcal{P},\mathcal{R},\Lambda).
$$

Proof. It suffices to note that the vector space isomorphism $\psi : \mathcal{P}/\mathcal{R} \ni [\pi] \to p_{\mathcal{S}}\pi \in \mathcal{S}$ intertwines the differentials $\overline{\partial}_\Lambda$ on $\mathcal{P}/\mathcal{R}$ and $\tilde{\partial}_\Lambda$ on $\mathcal{S}$. ∎

Remark. In view of this proposition it is clear that $\mathcal{S}$-cohomology is independent of the chosen splitting $\mathcal{P} = \mathcal{R} \oplus \mathcal{S}$.

Theorem 6. *The LP-cohomology groups of a SRMI Poisson tensor Λ, over the space $\mathcal{R}$ of cochains with coefficients in the formal power series, are given by*

$$
LH^{pr}(\mathcal{R},\Lambda) \simeq LH^{pr}(\mathcal{P},\Lambda)/\ker^{pr}\phi_\sharp \oplus LH^{p-1,r}(\mathcal{P},\mathcal{R},\Lambda)/\ker^{p-1,r}\phi_\sharp,
$$

where the above-introduced notations have been used.

Proof. Reformulation of Equation (8) and Proposition 3. ∎

Remark.*This theorem reduces computation of the formal LP-cohomology groups $LH^{pr}(\mathcal{R},\Lambda)$, basically to the Koszul cohomology groups $LH^{pr}(\mathcal{P},\Lambda) \simeq KH^p(\vec{X}_\delta, E_r)$ associated to the afore-detailed operators $\vec{X}_\delta$ on $E_r = \mathcal{S}^r\mathbb{R}^{n*}$ induced by the considered SRMI tensor, and to the relative cohomology groups $LH^{p-1,r}(\mathcal{P},\mathcal{R},\Lambda)$. It thus highlights the link between Poisson and Koszul cohomology. Let us mention that we showed in [MP06], via explicit computations in $\mathbb{R}^3$, that $\mathcal{P}$-cohomology (now identified as Koszul cohomology) and $\mathcal{S}$-cohomology (or relative cohomology) are less intricate than Poisson*

cohomology.

The remark concerning the comparative simplicity of the $\mathcal{P}$-cohomology can be easily understood.

Observe that any SRMI Poisson tensor $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ij} \in \mathbb{R}$, with $Y_i = \sum_r \ell_{ir}\partial_r$ and $\ell_{ir} \in \mathbb{R}^{n*}$, reads, locally in $\{D := \det \ell \neq 0\} \subset \{\Lambda \neq 0\} \subset \mathbb{R}^n$,

$$\Lambda = \sum_{i<j} \alpha^{ij} \partial_{s_i s_j} \ \ (\alpha^{ij} \in \mathbb{R}),$$

where $(s_1, \ldots, s_n)$ are local coordinates. As the Y_i mutually commute, the statement is a direct consequence of the "straightening theorem for vector fields". For instance, for structure $\Lambda = 2a\, Y_{23} + b\, Y_{31} + a\, Y_{12}$, where $Y_1 = x\partial_1 + y\partial_2$, $Y_2 = x\partial_2 - y\partial_1$, $Y_3 = z\partial_3$, and $D = (x^2+y^2)z$, see Theorem 2, the local (non-polynomial) coordinate transformation

$$x = e^s \cos\theta, y = e^s \sin\theta, z = -e^{-t}$$

leads to $Y_1 = \partial_s, Y_2 = \partial_\theta, Y_3 = \partial_t$, and $\Lambda = 2a\partial_{\theta t} + b\partial_{ts} + a\partial_{s\theta}$.

Hence, *locally* in a dense open subset of $\mathbb{R}^n$, there are coordinate systems or bases in which tensor Λ has *constant coefficients*. The $\mathcal{P}$-cohomology $LH^{*r}(\mathcal{P}, \Lambda)$ however, is the LP-cohomology in the extended space $\mathcal{P}^{*r} = D^{-1} \oplus_{p=0}^{n} \mathcal{S}^r\mathbb{R}^{n*} \otimes \wedge_n^p \langle \vec{Y} \rangle$, which admits the *global* basis $\vec{Y} = (Y_1, \ldots, Y_n)$ in which structure Λ has *constant coefficients*. This is what makes $\mathcal{P}$-cohomology particularly convenient.

4 Koszul cohomology in a finite-dimensional vector space

In view of the above remark regarding the basic ingredients of LP-cohomology of SRMI tensors of $\mathbb{R}^n$, we take in this section an interest in the Koszul cohomology space $KH^*(\vec{X}_\lambda, E)$ associated to operators $\vec{X}_\lambda := (X_1 - \lambda_1 \,\mathrm{id}, \ldots, X_n - \lambda_n \,\mathrm{id})$ made up of commuting linear transformations $\vec{X} := (X_1, \ldots, X_n)$ of a finite-dimensional real vector space E and a point $\vec{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. However, Koszul cohomology is known to be closely connected with Spectral Theory—a fundamental principle of multivariate operator theory is that all essential spectral properties of operators $\vec{X}$ in a complex space should be understood in terms of properties of the Koszul complex induced by $\vec{X}_\lambda$, $\vec{\lambda} \in \mathbb{C}^n$—so that the natural framework for investigations on Koszul cohomology is the complex setting.

Proposition 4. *Let (E, ∂) be a differential space over $\mathbb{R}$, and denote by $(E^{\mathbb{C}}, \partial^{\mathbb{C}})$ its complexification. The complexification $H^{\mathbb{C}}(E, \partial)$ of the cohomology space of (E, ∂) and the cohomology $H(E^{\mathbb{C}}, \partial^{\mathbb{C}})$ of differential space $(E^{\mathbb{C}}, \partial^{\mathbb{C}})$, are canonically isomorphic:*

$$H(E^{\mathbb{C}}, \partial^{\mathbb{C}}) \simeq H^{\mathbb{C}}(E, \partial).$$

Proof. Obvious. ∎

Proposition 5. *If $\vec{X} \in \mathrm{End}_{\mathbb{R}}(E)$ are commuting $\mathbb{R}$-linear transformations of a real vector space E, and if $\vec{X}^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}})$ are the commuting corresponding complexified $\mathbb{C}$-linear transformations of the complexification $E^{\mathbb{C}}$ of E, the following isomorphism of complex vector spaces holds:*

$$KH^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}}) \simeq KH^{*\mathbb{C}}(\vec{X}, E).$$

Proof. In view of Proposition 4, it suffices to check that the complex $K^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ is effectively the complexification of the complex $K^*(\vec{X}, E)$. ∎

This proposition allows deducing our subject for investigation, the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ (where $\vec{\lambda}$ is a point of $\mathbb{R}^n$ and where $\vec{X}$ is an n-tuple of commuting $\mathbb{R}$-linear operators of a finite-dimensional vector space E over $\mathbb{R}$), from its more natural counterpart over the field of complex numbers.

Below, we use the concept of joint spectrum $\sigma(\vec{X})$ of commuting bounded linear operators $\vec{X} = (X_1, \ldots, X_n)$ on a complex vector space E. There are a number of definitions of such spectra in the literature; the considered spaces E are normed spaces, Banach spaces, or Hilbert spaces. Here we investigate Koszul cohomology in finite dimension and need the following characterizations of the elements of the joint spectrum $\sigma(\vec{X})$ (for a proof, we refer the reader to [BR02]):

Proposition 6. *Let $\vec{X} = (X_1, \ldots, X_n)$ be an n-tuple of commuting operators on a finite-dimensional complex vector space E. Then the following statements are equivalent for any fixed $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$:*

(a) $\vec{\lambda} \in \sigma(\vec{X})$

(b) *There exists a basis in E with respect to which the matrices representing the X_j are all upper-triangular, and there exists an index q $(1 \leq q \leq \dim E)$, such that λ_j is the (q, q) entry of the matrix representing X_j, for $j \in \{1, \ldots, n\}$*

(c) *There exists an index q as in Item (b) for every basis in E with respect to which the matrices representing the X_j are all upper-triangular*

(d) *There exists a nonzero vector x such that $X_j x = \lambda_j x$, $\forall j \in \{1, \ldots, n\}$*

(e) *There do not exist Y_j in the subalgebra of $\mathrm{End}_{\mathbb{C}}(E)$ generated by id and $\vec{X}$, such that*

$$\sum_{j=1}^{n} Y_j (X_j - \lambda_j \,\mathrm{id}) = \mathrm{id}$$

In the following, we supply some results regarding Koszul cohomology spaces. We use the same notations as above.

Proposition 7. *Let $\wedge = \wedge_n \langle \vec{\eta} \rangle$ be the exterior algebra on n generators $\vec{\eta}$ over a field $\mathbb{F}$ of characteristic 0, and let $\vec{h}$ be dual generators, i.e. $i_{h_k} \eta_\ell = \partial_{k\ell}$. We then have the homotopy formula*

$$e_{\eta_\ell} i_{h_k} + i_{h_k} e_{\eta_\ell} = \delta_{k\ell} \,\mathrm{id},$$

where i_{h_k} and e_{η_ℓ} are the creation and annihilation operators, respectively.

Proof. Obvious. ∎

Proposition 8. *Let $\vec{\mathcal{X}} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ (resp. $\vec{Y} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$) be n commuting linear operators $\vec{\mathcal{X}}$ (resp. $\vec{Y}$) on a vector space E over $\mathbb{F}$. We denote by $\mathcal{K} = \sum_\ell \mathcal{X}_\ell \otimes e_{\eta_\ell}$ (resp. $\kappa = \sum_k Y_k \otimes i_{h_k}$) the corresponding Koszul cohomology (resp. homology) operator. The following homotopy-type result holds:*

$$\mathcal{K}\kappa + \kappa\mathcal{K} = \left(\sum_\ell Y_\ell \mathcal{X}_\ell \right) \otimes \mathrm{id} + \sum_{k\ell} [\mathcal{X}_\ell, Y_k] \otimes e_{\eta_\ell} i_{h_k}.$$

Proof. Direct consequence of Proposition 7. ∎

Proposition 9. *Let $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space E, and let $\vec{\lambda} \in \mathbb{C}^n$. Consider a splitting*

$$E = E_1 \oplus E_2$$

and denote by $i_j : E_j \to E$ (resp. $p_j : E \to E_j$) the injection of E_j into E (resp. the projection of E onto E_j).

If E_1 is stable under the action of the operators X_ℓ, i.e. $p_2 \, X_\ell \, i_1 = 0$, and if $\vec{\lambda}$ is not in the joint spectrum $\sigma(\vec{X}_2)$ of the commuting operators $X_{\ell 2} = p_2 \, X_\ell \, i_2 \in \mathrm{End}_{\mathbb{C}}(E_2)$, then any cocycle $C \in E \otimes \wedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$, where $\vec{X}_\lambda = \vec{X} - \vec{\lambda} \,\mathrm{id}_E$, is cohomologous to a cocycle $C_1 \in E_1 \otimes \wedge$, with $\wedge = \wedge_n \langle \vec{\eta} \rangle$.

Proof. Observe first that if $q(\vec{X}) \in \mathbb{C}[X_1, \ldots, X_n] \subset \operatorname{End}_{\mathbb{C}}(E)$ denotes a polynomial in the X_ℓ, the compound map $q(\vec{X})_2 = p_2 q(\vec{X}) i_2$ coincides with the (same) polynomial $q(\vec{X}_2) \in \operatorname{End}_{\mathbb{C}}(E_2)$ in the $X_{\ell 2}$ $(\star)$. Indeed, due to stability of E_1, we have

$$p_2\, X_\ell X_k\, i_2 = p_2\, X_\ell\, i_1\, p_1\, X_k\, i_2 + p_2\, X_\ell\, i_2\, p_2\, X_k\, i_2 = X_{\ell 2} X_{k2}.$$

This entails in particular that the $X_{\ell 2}$ commute.

As $\vec{\lambda} \notin \sigma(\vec{X}_2)$, Item (e) in Proposition 6 implies that there are n operators $\vec{Y}_2$ in the subalgebra of $\operatorname{End}_{\mathbb{C}}(E_2)$ generated by $\mathrm{id}_{\mathrm{E}_2}$ and $\vec{X}_2$, such that

$$\sum_\ell Y_{\ell 2}(X_{\ell 2} - \lambda_\ell \, \mathrm{id}_{E_2}) = \mathrm{id}_{E_2}\,. \tag{9}$$

Hence, for any ℓ, $Y_{\ell 2} = Q_\ell(\vec{X}_2)$ is a polynomial in the X_{k2}. Set now $Y_\ell = Q_\ell(\vec{X}) \in \operatorname{End}_{\mathbb{C}}(E)$.

If applied to operators $\vec{X}_\lambda$ and $\vec{Y}$, Proposition 8 implies that

$$\left(\sum_\ell Y_\ell(X_\ell - \lambda_\ell \, \mathrm{id}_E)\right) \otimes \mathrm{id}_\wedge + \sum_{k\ell} [X_\ell - \lambda_\ell \, \mathrm{id}_E, Y_k] \otimes e_{\eta_\ell} i_{h_k} = \mathcal{K}\kappa + \kappa\mathcal{K},$$

where $\mathcal{K}$ (resp. κ) is the Koszul cohomology (resp. homology) operator associated with $\vec{X}_\lambda$ (resp. $\vec{Y}$) on E. As Y_k is a polynomial in the commuting endomorphisms X_ℓ, the second term on the LHS of the preceding equation vanishes. Hence, when evaluating both sides on a cocycle $C = e \otimes w$ of cochain complex $K^*(\vec{X}_\lambda, E)$, we get

$$\left(Q(\vec{X})(e)\right) w = \mathcal{K}\kappa(e \otimes w),$$

where $Q(\vec{X}) = \sum_\ell Y_\ell(X_\ell - \lambda_\ell \, \mathrm{id}_E) = \sum_\ell Q_\ell(\vec{X})(X_\ell - \lambda_\ell \, \mathrm{id}_E)$ is a polynomial in the X_ℓ. Up to factor w, the LHS reads

$$Q(\vec{X})(e) = p_1 Q(\vec{X}) i_1 p_1(e) + p_2 Q(\vec{X}) i_1 p_1(e) + p_1 Q(\vec{X}) i_2 p_2(e) + p_2 Q(\vec{X}) i_2 p_2(e),$$

where the second term of the RHS vanishes, in view of the stability of E_1, and where the last term coincides with $p_2(e)$, in view of Remark $(\star)$ and Equation (9). Eventually, cocycle $C = e \otimes w$ is cohomologous to cocycle

$$C_1 = C - \mathcal{K}\kappa C = \left(p_1(e) - p_1 Q(\vec{X}) i_1 p_1(e) - p_1 Q(\vec{X}) i_2 p_2(e)\right) \otimes w \in E_1 \otimes \wedge. \blacksquare$$

The preceding proposition allows in particular recovering the following well-known result:

Corollary 2. *Consider n commuting endomorphisms $\vec{X} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E)$ of a finite-dimensional complex vector space E, and a point $\vec{\lambda} \in \mathbb{C}^n$. Set $\ker \vec{X}_\lambda := \cap_{\ell=1}^n \ker(X_\ell - \lambda_\ell \, \mathrm{id})$. If $\dim(\ker \vec{X}_\lambda) = 0$, the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ is trivial, and vice versa.*

Proof. It suffices to note that, due to Proposition 6, the dimensional assumption means that $\vec{\lambda} \notin \sigma(\vec{X})$, and to apply the preceding proposition with $E_1 = 0$. Conversely, if there exists $x \in \ker \vec{X}_\lambda \setminus \{0\}$, then $\mathcal{K}_{\vec{X}_\lambda} x = \sum_{\ell=1}^n (X_\ell - \lambda_\ell \, \mathrm{id})(x)\, \eta_\ell = 0$, so that x is a nonbounding 0-cocycle. $\blacksquare$

The next consequence of Proposition 9 shows that the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ is—roughly spoken—made up by joint eigenvectors with eigenvalues λ_ℓ.

Consider n commuting endomorphisms $\vec{X} =: \vec{X}^{(1)} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E)$ of a finite-dimensional complex vector space $E =: E^{(1)} =: F^{(1)}$, and a point $\vec{\lambda} \in \mathbb{C}^n$. For any $a \in \{2, 3, \ldots\}$, if $\ker^{(a-1)} := \ker \vec{X}_\lambda^{(a-1)}$ and $E^{(a)} := E^{(a-1)} / \ker^{(a-1)}$, the

$$X_\ell^{(a)} := \left(X_\ell^{(a-1)}\right)^\sharp,$$

$\ell \in \{1, \ldots, n\}$, defined recursively by $X_\ell^{(a)} = \left(X_\ell^{(a-1)}\right)^\sharp : E^{(a)} \ni [e^{(a-1)}] \to [X_\ell^{(a-1)} e^{(a-1)}] \in E^{(a)}$, are again n commuting (well-defined) operators on a finite-dimensional complex vector space. We iterate this procedure finitely many times, thus obtaining operators $X_\ell^{(a)}$, $a \in \{1, \ldots, s+1\}$, until $\ker^{(s+1)} = \ker \vec{X}_\lambda^{(s+1)} = 0$, or, equivalently,

$$\vec{\lambda} \notin \sigma(\vec{X}^{(s+1)}).$$

In the following, we identify the operators $X_\ell^{(a)}$ with their models that arise from the choices of supplementary subspaces $F^{(a)}$ of $\ker^{(a-1)}$ in $E^{(a-1)} \simeq F^{(a-1)}$, $a \in \{2, \ldots, s+1\}$, so that $E^{(a)} \simeq F^{(a)} \subset E^{(a-1)} \simeq F^{(a-1)}$. If we denote by $i_a : F^{(a)} \to F^{(a-1)}$ the inclusion and by $p_a : F^{(a-1)} \to F^{(a)}$ the canonical projection, the isomorphism $E^{(a)} \simeq F^{(a)}$ is $E^{(a)} \ni [f^{(a-1)}] \leftrightarrow p_a f^{(a-1)} \in F^{(a)}$, and operator $X_\ell^{(a)}$, viewed as endomorphism of $F^{(a)}$, reads

$$X_\ell^{(a)} = p_a X_\ell^{(a-1)} i_a, \tag{10}$$

since for any $f^{(a)} \in F^{(a)}$, we have $X_\ell^{(a)} f^{(a)} = X_\ell^{(a)}[f^{(a)}] = [X_\ell^{(a-1)} f^{(a)}] = p_a X_\ell^{(a-1)} i_a f^{(a)}$.

Corollary 3. *Let $\vec{\lambda} \in \mathbb{C}^n$ be a point in $\mathbb{C}^n$, and let $\vec{X} = \vec{X}^{(1)} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space $E = F^{(1)}$. Denote by $\vec{X}^{(a)} \in \mathrm{End}_{\mathbb{C}}^{\times n}(F^{(a)})$, $a \in \{2, \ldots, s\}$, the above-depicted "reduced" operators on supplementary spaces $F^{(a)}$, and denote by $\wedge = \wedge_n\langle \vec{\eta} \rangle$ the Grassmann algebra with n generators $\vec{\eta}$.*

Any cocycle

$$C \in E \otimes \wedge$$

of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle

$$C_1 \in \left(\ker \vec{X}_\lambda^{(1)} \oplus \ker \vec{X}_\lambda^{(2)} \oplus \ldots \oplus \ker \vec{X}_\lambda^{(s)}\right) \otimes \wedge.$$

Proof. It suffices to apply Proposition 9 to the obvious splitting

$$E = E_1 \oplus E_2 := \left(\bigoplus_{a=1}^{s} \ker^{(a)}\right) \oplus F^{(s+1)}.$$

Indeed, the operators $\vec{X}_2$ considered in Proposition 9 read $X_{\ell 2} = p_{s+1} \ldots p_2 X_\ell i_2 \ldots i_{s+1} = X_\ell^{(s+1)}$, where we used the afore-introduced notations i_a and p_a. Hence, the spectral condition $\vec{\lambda} \notin \sigma(\vec{X}_2)$ is satisfied by definition of s, see above. Moreover, if $k^{(a)} \in \ker^{(a)} \subset F^{(a)}$, $a \in \{1, \ldots, s\}$, we have

$$X_\ell\, k^{(a)} = X_\ell\, i_2 \ldots i_a\, k^{(a)} = p_a \ldots p_2\, X_\ell\, i_2 \ldots i_a\, k^{(a)} + \sum_{b=2}^{a} \pi_b\, p_{b-1} \ldots p_2\, X_\ell\, i_2 \ldots i_a\, k^{(a)}. \tag{11}$$

Mapping $\pi_b : F^{(b-1)} \to \ker^{(b-1)}$ is the second projection associated with the decomposition $F^{(b-1)} = F^{(b)} \oplus \ker^{(b-1)}$, so that $\mathrm{id}_{F^{(b-1)}} = p_b + \pi_b$. In order to derive Equation (11), we utilized this upshot for $b \in \{2, \ldots, a\}$. The first term of the RHS of Equation (11) is $X_\ell^{(a)} k^{(a)} = \lambda_\ell\, k^{(a)} \in \ker^{(a)}$, and the terms characterized by index b are elements of the spaces $\ker^{(b-1)}$. Hence, space $E_1 = \oplus_{a=1}^s \ker^{(a)}$ is stable under the action of the X_ℓ and Proposition 9 is applicable. ∎

Corollary 4. *On the conditions of Corollary 3, if for any $\ell \in \{1, \ldots, n\}$, the kernel and the image of the transformation $X_\ell - \lambda_\ell \,\mathrm{id}$ are supplementary in E, then any cocycle $C \in E \otimes \wedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \ker \vec{X}_\lambda \otimes \wedge$.*

Proof. It suffices to prove that $s = 1$. If $s \neq 1$, there is a nonzero vector $x \in \ker \vec{X}_\lambda^{(2)} \subset F^{(2)}$. Then, for any $k, \ell \in \{1, \ldots, n\}$, $(X_k - \lambda_k \,\mathrm{id})(X_\ell - \lambda_\ell \,\mathrm{id})x = (X_k - \lambda_k \,\mathrm{id})(p_2 X_\ell i_2 x + \pi_2 X_\ell i_2 x - \lambda_\ell x) = (X_k - \lambda_k \,\mathrm{id})(\pi_2 X_\ell i_2 x) = 0$, as $\pi_2 X_\ell i_2 x \in \ker \vec{X}_\lambda$. Hence, for every ℓ, we have $(X_\ell - \lambda_\ell \,\mathrm{id})x \in \ker \vec{X}_\lambda \cap \mathrm{im}(X_\ell - \lambda_\ell \,\mathrm{id}) = 0$. Eventually, $x \in (\ker \vec{X}_\lambda) \cap F^{(2)} = 0$, a contradiction.

5 Koszul cohomology associated with Poisson cohomology

We now come back to the Koszul cohomology implemented by a SRMI tensor of $\mathbb{R}^n$. Let us recall that we deal with a SRMI tensor

$$\Lambda = \sum_{j<k} \alpha^{jk} Y_{jk} \quad (\alpha^{jk} \in \mathbb{R}),$$

where the Y_j are n commuting linear vector fields that verify $Y_{1\ldots n} \neq 0$. The main building block of the LP-cohomology of such a tensor has been identified as the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$ associated to the operators $\vec{X}_\delta = (X_1 - \delta_1 \,\mathrm{id}, \ldots, X_n - \delta_n \,\mathrm{id})$, $X_j = \sum_k \alpha^{jk} Y_k$, $\alpha^{kj} = -\alpha^{jk}$, $\delta_j = \operatorname{div} X_j$ on the spaces $E_r = \mathcal{S}^r \mathbb{R}^{n*}$, $r \in \mathbb{N}$. We already pointed out that this cohomology can be deduced from its complex counterpart $KH^*(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_r)$ (see Proposition 5), which is tightly related with joint eigenvectors and the joint spectrum of $\vec{X}^{\mathbb{C}}$ or $\vec{X}^{\mathbb{C}}_\delta$ (see Corollaries 2 and 3). In this section, we further investigate the Koszul cohomology space $KH^*(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_r)$. In particular, we reduce the computation of this central part of the LP-cohomology space $LH^{*r}(\mathcal{R}, \Lambda)$ to essentially a problem of linear algebra, and give a description of the spectrum of the transformations $\vec{X}^{\mathbb{C}}_\delta$.

When dealing with commuting operators on a finite-dimensional complex vector space, it is natural to use an upper-triangular representation of these transformations. The following theorem shows that, for our endomorphisms $\vec{X}^{\mathbb{C}}_\delta$ of the space $E^{\mathbb{C}}_r = \mathcal{S}^r \mathbb{C}^{n*}$ (see below), which has the possibly high (complex) dimension $N = (r+n-1)!/[r!(n-1)!]$ (if e.g. $r = 10$ and $n = 3$, this dimension equals $N = 66$), the problem of finding such a representation $\vec{X}^{\mathbb{C}}_\delta \in \mathrm{gl}(N, \mathbb{C})^{\times n}$ (we denote the operators and their representation by the same symbol) reduces to the quest for an upper-triangular representation $\vec{a} \in \mathrm{gl}(n, \mathbb{C})^{\times n}$ of some commuting transformations $\vec{a}$ of $\mathbb{C}^n$. More precisely, the a_j, $j \in \{1, \ldots, n\}$, are the commuting matrices $a_j = (J^1)^{-1} Y_j \in \mathrm{gl}(n, \mathbb{R})$ that correspond to the commuting linear vector fields Y_j.

Proposition 10. *Any basis of $\mathbb{C}^n$, in which the commuting operators $\vec{a}$ have an upper-triangular representation, naturally induces a basis of $E^{\mathbb{C}}_r = \mathcal{S}^r \mathbb{C}^{n*}$, in which all the transformations $\vec{X}^{\mathbb{C}}_\delta$ are upper-triangular.*

Let us first mention that in the sequel the use of super- and subscripts is dictated by esthetic criteria and not at all by contra- or covariance.

Proof. In the following, we denote by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$) the points of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) as well as their coordinates in the canonical basis $(e_1, \ldots, e_n)$. As usual, we set $Y_k = \sum_m \ell_{km} \partial_{x_m} = \sum_{mp} a^{mp}_k \, x_p \partial_{x_m}$ and use notations as $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$, $\beta \in \mathbb{N}^n$.

The complexification $E^{\mathbb{C}}_r$ of

$$E_r = \mathcal{S}^r \mathbb{R}^{n*} = \{P \in C^\infty(\mathbb{R}^n) : P(x) = \sum_{|\beta|=r} r_\beta x^\beta \quad (x \in \mathbb{R}^n, r_\beta \in \mathbb{R})\}$$

is

$$E_r \oplus iE_r \simeq E^{\mathbb{C}}_r \simeq \mathcal{S}^r \mathbb{C}^{n*} = \{P \in C^\infty(\mathbb{C}^n) : P(z) = \sum_{|\beta|=r} c_\beta z^\beta \quad (z \in \mathbb{C}^n, c_\beta \in \mathbb{C})\}.$$

It is also easily seen that the complexification $Y^{\mathbb{C}}_k \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}}_r)$ of $Y_k \in \mathrm{End}_{\mathbb{R}}(E_r)$ is the holomorphic vector field

$$Y^{\mathbb{C}}_k = \sum_{mp} a^{mp}_k \, z_p \partial_{z_m} \in \mathrm{Vect}^{10}(\mathbb{C}^n)$$

of $\mathbb{C}^n$.

It is well-known that the n commuting matrices $a_j = (J^1)^{-1} Y_j \in \mathrm{gl}(n, \mathbb{R})$ can be reduced simultaneously to upper-triangular matrices by a unitary matrix $U \in \mathrm{U}(n, \mathbb{C})$. Consider any matrix $U \in \mathrm{GL}(n, \mathbb{C})$ (resp. any basis $(e'_1, \ldots, e'_n)$ of $\mathbb{C}^n$), such that the $b_j = U^{-1} a_j U \in \mathrm{gl}(n, \mathbb{C})$ are upper-triangular (resp. in which the transformations $\vec{a}$ are all upper-triangular). Denote by $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_n)$

the components of the vectors $z = \sum_j \mathfrak{z}_j e'_j \in \mathbb{C}^n$ in the basis $(e'_1, \ldots, e'_n)$, and let $(\varepsilon'_1, \ldots, \varepsilon'_n)$ be the dual basis of this new basis. If viewed as a basis of the space $E_r^{\mathbb{C}}$ of degree r homogeneous polynomials of $\mathbb{C}^n$, the induced basis $\varepsilon'_{j_1} \vee \ldots \vee \varepsilon'_{j_r}$, $j_1 \le \ldots \le j_r$, of the space $\mathcal{S}^r\mathbb{C}^{n*}$ of symmetric covariant r-tensors of $\mathbb{C}^n$ reads $\mathfrak{z}^\beta$, $\beta \in \mathbb{N}^n$, $|\beta| = r$.

In order to find the matrices of the operators $\vec{X}_\delta^{\mathbb{C}}$ in this "natural" basis $\mathfrak{z}^\beta$, $\beta \in \mathbb{N}^n$, $|\beta| = r$ of $E_r^{\mathbb{C}}$, we range the vectors $\mathfrak{z}^\beta$ according to the lexicographic order $\prec$ and perform the coordinate change $z = U\mathfrak{z}, \partial_z = \widetilde{\partial_{\mathfrak{z}} z}^{-1} \partial_{\mathfrak{z}}$ in the first order linear differential operators $(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}}$. We get

$$\begin{aligned}(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} &= \sum_k \alpha^{jk} \sum_{m \le p} b_k^{mp}\, \mathfrak{z}_p \partial_{\mathfrak{z}_m} - \delta_j \,\mathrm{id}^{\mathbb{C}} \\ &= \sum_{km} \alpha^{jk} b_k^{mm} \left(\mathfrak{z}_m \partial_{\mathfrak{z}_m} - \mathrm{id}^{\mathbb{C}}\right) + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp}\, \mathfrak{z}_p \partial_{\mathfrak{z}_m},\end{aligned}$$

since $\delta_j = \mathrm{div}\, X_j = \sum_{km} \alpha^{jk} a_k^{mm} = \sum_{km} \alpha^{jk} b_k^{mm}$. As the image of vector $\mathfrak{z}^\beta$ by operator $(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}}$ is

$$(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} \mathfrak{z}^\beta = \sum_{km} \alpha^{jk} b_k^{mm}\, (\beta_m - 1)\, \mathfrak{z}^\beta + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \beta_m\, \mathfrak{z}^{\beta - e_m + e_p}, \tag{12}$$

where $\mathfrak{z}^{\beta - e_m + e_p} \prec \mathfrak{z}^\beta$, the matrices of the commuting operators $(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}}$, $j \in \{1, \ldots, n\}$, in the basis $\mathfrak{z}^\beta$, $\beta \in \mathbb{N}^n$, $|\beta| = r$, of space $E_r^{\mathbb{C}}$, are all upper-triangular. ∎

The next theorem provides a description of the joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the operators $\vec{X}_\delta^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E_r^{\mathbb{C}})$.

Let $B \in \mathrm{gl}(n, \mathbb{C})$ be the matrix $B_{jk} = b_j^{kk}$ made up by the diagonals of the matrices b_j, see above.

Theorem 7. *The joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the commuting operators $\vec{X}_\delta^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E_r^{\mathbb{C}})$ on the finite-dimensional complex vector space $E_r^{\mathbb{C}}$, is given by*

$$\sigma_r(\vec{X}_\delta^{\mathbb{C}}) = \{\alpha B I : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n\} \subset \mathbb{C}^n,$$

where $|I| = \sum_j I_j$ denotes the length of I.

Proof. Direct consequence of Proposition 6 and Equation (12). ∎

Remark. In Proposition 1, we showed that for all k, $Y_k D = (\mathrm{div}\, Y_k) D$, where $D = \det \ell \in E_n \subset E_n^{\mathbb{C}}$. It of course follows that for all j, $X_j^{\mathbb{C}} D = X_j D = (\mathrm{div}\, X_j) D = \delta_j \,\mathrm{id}^{\mathbb{C}} D$, so that $\vec{0} = (0, \ldots, 0) \in \sigma_n(\vec{X}_\delta^{\mathbb{C}})$. This last upshot is immediately recovered from Theorem 7.

Set $K_r(\vec{X}_\delta^{\mathbb{C}}) = \{I \in \ker(\alpha B) : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n\}$. Corollary 2 can then be reformulated as follows.

Corollary 5. *The Koszul cohomology $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is acyclic if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$.*

Proof. Indeed, $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is trivial if and only if $\dim(\ker \vec{X}_\delta^{\mathbb{C}}) = 0$, if and only if $\vec{0} \notin \sigma_r(\vec{X}_\delta^{\mathbb{C}})$, i.e. if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$. ∎

We now depict a convenient method that allows finding a basis of the space

$$\ker \vec{X}_\delta^{\mathbb{C}\,(1)} \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(2)} \oplus \ldots \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(s)},$$

which houses the Koszul cohomology $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$, see Corollary 3.

In order to simplify notations, we systematically omit in the following description superscript $\mathbb{C}$. We write e.g. $\vec{X}_\delta, E_r, \ldots$ instead of $\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}}, \ldots$

Consider any basis $(e_1,\ldots,e_N)$ of E_r that generates an upper-triangular representation $T_1,\ldots,T_n$ of the operators $\vec{X}_\delta$. The kernel $\ker\vec{X}_\delta$ is then described by the n triangular systems

$$T_1Z=0,\ldots,T_nZ=0, \tag{13}$$

(each one) of N homogeneous linear equations in the N complex unknowns $Z=(Z^1,\ldots,Z^N)$.

As understood before, $\vec{0}\in\sigma_r(\vec{X}_\delta)$ if and only if at least one of the lines $\vec{T}^{\,q}=(T_1^{qq},\ldots,T_n^{qq})$, $q\in\{1,\ldots,N\}$, is $\vec{0}=(0,\ldots,0)$. We refer to the number μ of such $\vec{0}$–lines $\vec{T}^{\,q_1},\ldots,\vec{T}^{\,q_\mu}$, $q_1<\ldots<q_\mu$, as the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}_\delta)$ (in the considered basis $(e_1,\ldots,e_N)$). Of course, the general solution of System (13) is a linear combination $Z=\sum_j c_jK_j$, $c_j\in\mathbb{C}$, of $d=\dim\ker\vec{X}_\delta$ independent vectors $K_j\in\mathbb{C}^N$. Let

$$k_j=K_j^1e_1+\ldots+K_j^{q_{\nu_j}}e_{q_{\nu_j}},\ j\in\{1,\ldots,d\}, \tag{14}$$

be the corresponding basis of $\ker\vec{X}_\delta$. It can quite easily be seen—just "solve" System (13) and start imagining a configuration that leads to the maximal dimension of the space of solutions—that $d\le\mu$ and that the components $K_j^{q_{\nu_j}}\neq 0$ of the vectors k_j with highest superscript correspond to $\vec{0}$–lines $q_{\nu_1}<\ldots<q_{\nu_d}$.

The N-tuple $(k_1,\ldots,k_d,e_1,\ldots,\widehat{e_{q_{\nu_1}}},\ldots,\widehat{e_{q_{\nu_d}}},\ldots,e_N)$ is a basis of E_r, since the determinant in the basis $(e_1,\ldots,e_N)$ of the permuted N-tuple $(e_1,\ldots,k_1,\ldots,k_d,\ldots,e_N)$ equals $K_1^{q_{\nu_1}}\ldots K_d^{q_{\nu_d}}\neq 0$. Observe that the k_j are joint eigenvectors of the $\vec{X}_\delta$ associated with eigenvalue 0. Moreover, in view of Equation (14), every vector $e_{q_{\nu_j}}$ can be written in terms of "lower" vectors of the new basis. Hence, the first d columns of the representative matrices $T'_1,\ldots,T'_n$ of the operators $\vec{X}_\delta$ in the new basis vanish, these matrices are again upper-triangular, and the lines $\vec{T}^q$, $q\in\{1,\ldots,N\}$, are unchanged up to permutation. The matrices $T'_\ell+\delta_\ell\,\mathrm{id}\in\mathrm{gl}(N,\mathbb{C})$ correspond to the operators X_ℓ, $\ell\in\{1,\ldots,n\}$, and their lower right submatrices $(T'_\ell+\delta_\ell\,\mathrm{id})^{(2)}\in\mathrm{gl}(N-d,\mathbb{C})$ (resp. $T'^{(2)}_\ell$) correspond to the operators $X_\ell^{(2)}$ (resp. $X_\ell^{(2)}-\delta_\ell\,\mathrm{id}^{(2)}$), see Equation (10) and Corollary 3.

In other words, in the basis $(e_1,\ldots,\widehat{e_{q_{\nu_1}}},\ldots,\widehat{e_{q_{\nu_d}}},\ldots,e_N)$ of a space $F_r^{(2)}$, see Corollary 3, which is supplementary to $\ker\vec{X}_\delta$ in E_r, the operators $\vec{X}_\delta^{(2)}$ are represented by upper-triangular matrices $T'^{(2)}_1,\ldots,T'^{(2)}_n$. Thus, the above-detailed procedure can be iterated and the general solution of another packet of n (smaller) triangular systems of linear equations

$$T'^{(2)}_1Z=0,\ldots,T'^{(2)}_nZ=0, \tag{15}$$

provides a basis $k_1^{(2)},\ldots,k_{d_2}^{(2)}$ of $\ker\vec{X}_\delta^{(2)}$, et cetera.

Remarks.

- *The solutions of the triangular systems of homogeneous linear equations (13), (15), ... generate a basis of the locus*

  $$\ker\vec{X}_\delta^{\mathbb{C}\,(1)}\oplus\ker\vec{X}_\delta^{\mathbb{C}\,(2)}\oplus\ldots\oplus\ker\vec{X}_\delta^{\mathbb{C}\,(s)}$$

  of the Koszul cohomology space $KH^*(\vec{X}_\delta^{\mathbb{C}},E_r^{\mathbb{C}})$.

- Observe that if the $b_\ell=U^{-1}a_\ell U$ have been computed, the upper-triangular matrix representations $T_1,\ldots,T_n$ of the transformations $\vec{X}_\delta^{\mathbb{C}}$ in the corresponding basis $\mathfrak{z}^\beta$, $\beta\in\mathbb{N}^n$, $|\beta|=r$, of $E_r^{\mathbb{C}}$ are known, see Equation (12), and explicit computations can actually be performed.

- As the multiplicity of $\vec{0}$ in the spectrum of the endomorphisms $\vec{X}_\delta^{\mathbb{C}\,(2)}$ is $\mu-d$, and as its multiplicity in the spectrum of the $\vec{X}_\delta^{\mathbb{C}\,(s+1)}$ vanishes, by definition of s, we get

$$\mu=d+d_2+\ldots+d_s=\sum_{j=1}^{s}\dim\ker\vec{X}_\delta^{\mathbb{C}\,(j)}, \tag{16}$$

with self-explaining notations. As the RHS of this equation is independent of the considered basis, the concept of multiplicity of a point $\lambda \in \mathbb{C}^n$ in the joint spectrum of commuting transformations of a finite-dimensional vector space, makes sense. Although this result might be well-known, we could not find it anywhere in literature.

Example 1. Consider structure Λ_2 of the DHC, see Theorem 2, and assume that $a \neq 0, b = 0$. It is easily checked that the matrix

$$U = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}$$

transforms the above-mentioned matrices a_ℓ simultaneously into upper-triangular matrices b_ℓ. A short computation yields that the space $K_{3t}(\vec{X}^{\mathbb{C}}_\delta)$, $t \in \mathbb{N}$, contains the unique point $I_t = (t-1, t-1, t-1)$, so that the multiplicity μ of $\vec{0}$ in the joint spectrum $\sigma_{3t}(\vec{X}^{\mathbb{C}}_\delta)$ equals 1, see proof of Theorem 7. It follows that the Koszul cohomology spaces $KH^*(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_{3t})$ are not trivial, see Corollary 5. Furthermore, since the matrices b_ℓ are in fact diagonal in this example, Equation (12) entails that $\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ belongs to the kernel $\ker_{3t} \vec{X}^{\mathbb{C}}_\delta$ of operators $\vec{X}^{\mathbb{C}}_\delta$ in space $E^{\mathbb{C}}_{3t}$. If we take into account Equation (16), we see that $\ker_{3t} \vec{X}^{\mathbb{C}}_\delta = \mathbb{C}\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ and that the reduced operators $\vec{X}^{\mathbb{C}\,(j)}_\delta$, $j \in \{2, \ldots, s\}$, do not exist, i.e. that $s = 1$. Hence, and since the change to canonical coordinates is $z = U\mathfrak{z}$, see proof of Proposition 10, the cohomology space $KH^p(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_{3t})$, $p \in \{0, 1, 2, 3\}$, $t \in \mathbb{N}$, is located inside

$$\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p} = (z_1^2 + z_2^2)^t z_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p}.$$

This rather easily obtained upshot is in accordance with the results of [MP06] (modulo slight changes in definitions and notations [e.g. the roles of parameters a and b are exchanged]).

Example 2. For structure Λ_3 of the DHC and parameter value $a = 0$, depending on the value of r, the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}^{\mathbb{C}}_\delta)$ equals 0 or 1—and computations are similar to those of the preceding example—, except in the case $r = 3$, which generates multiplicity 3. Since for Λ_3 the matrices a_ℓ are lower-triangular, a coordinate change $z \leftrightarrow \mathfrak{z}$ is not necessary and it can easily be seen that we have $s = 3$ and

$$\ker_3 \vec{X}^{\mathbb{C}}_\delta = \mathbb{C}z_1^2 z_3, \ker_3 \vec{X}^{\mathbb{C}\,(2)}_\delta = \mathbb{C}z_1 z_2 z_3, \ker_3 \vec{X}^{\mathbb{C}\,(3)}_\delta = \mathbb{C}z_2^2 z_3.$$

The corresponding cohomological upshots are part of the computation of the LP-cohomology of Λ_3 that we detail in the next section.

Remark. Remember that the operators X_i are defined by $X_i = \sum_j \alpha^{ij} Y_j$, with $\alpha^{ji} = -\alpha^{ij}$. Hence, matrix $\alpha \in \mathrm{gl}(n, \mathbb{R})$ is skew-symmetric, and $\det \alpha$ vanishes for odd n. Of course, the corresponding non-trivial linear combination $\sum_i c_i \alpha^{i*} = 0$, induces a non-trivial combination $\sum_i c_i X_i = 0$ of the X_i (and the $X_i - \delta_i\,\mathrm{id}$), which is significant in computations. In the even dimensional ($n = 2m, m \in \{2, 3, \ldots\}$) maximal rank ($\mathrm{rk}\,\alpha = n$) case, the Koszul cohomology $KH^*(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_r)$ has the following simple description. If (in even dimension n) $\det \alpha \neq 0$, then

$$\bigoplus_{r\in\mathbb{N}} KH^0(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_r) = \mathbb{C}\,\mathcal{D},$$

where $\mathcal{D}$ denotes the complex clone of $\det \ell$, and, for any $r \neq n$ and any $p \in \{1, \ldots, n\}$, the cohomology space $KH^p(\vec{X}^{\mathbb{C}}_\delta, E^{\mathbb{C}}_r)$ vanishes. We do not detail the proof that is, roughly, along the lines of Proposition 1. If $r = n$, the situation is more complicated and new elements of $\ker \vec{X}^{\mathbb{C}\,(1)}_\delta \oplus \ker \vec{X}^{\mathbb{C}\,(2)}_\delta \oplus \ldots \oplus \ker \vec{X}^{\mathbb{C}\,(s)}_\delta$ may enter the play.

6 Cohomology spaces of structures Λ_3 and Λ_9

We already pointed out that the LP-cohomology (or $\mathcal{R}$-cohomology) of SRMI tensors can be deduced from a Koszul cohomology ($\mathcal{P}$-cohomology) and a relative cohomology ($\mathcal{S}$-cohomology), see Theorem 6, Theorem 5, and Proposition 3.

The involved Koszul cohomology has been studied in the last section. We particularized our upshots by means of (pertinent) examples, see Examples 1 and 2, Section 5.

Within the cohomology computations of SRMI tensors of the DHC, $\mathcal{S}$-cohomology has so far been determined "by hand". In the majority of cases, the LP-cohomology operator respects, in addition to the degrees p and r, a partial polynomial degree k (e.g. the coboundary operator associated with Λ_3 respects the partial degree in $x = x_1, y = x_2$), so that we can decompose space $\mathcal{S}^{pr}$ into smaller spaces $\mathcal{S}^p_{kr}$ (made up by the elements of $\mathcal{S}^{pr}$ that have partial degree k), see [MP06]. The cohomology operator of structure Λ_9 however, does not respect any additional degree. The $\mathcal{S}$-cohomology of Λ_9 is therefore quite intricate.

Theorem 6 leads to the following cohomological upshots for structures Λ_3 and Λ_9. No proofs will be given (for a description of an application of the technique, see [MP06]).

Theorem 8. *If $a \neq 0$, the cohomology spaces of structure Λ_3 are*

$$LH^{0*}(\mathcal{R}, \Lambda_3) = \mathbb{R},$$

$$\begin{aligned} LH^{1*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3, \\ LH^{2*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}Y_{23} \oplus \mathbb{R}Y_{31} \oplus \mathbb{R}(2yz\partial_{31} + y^2\partial_{12}), \\ LH^{3*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}\,\partial_{123} \oplus \mathbb{R}\,y^2 z\,\partial_{123}, \end{aligned}$$

where the Y_i are those defined in Theorem 2.

Theorem 9. *If $a \neq 0$, the cohomology spaces of structure Λ_9 are*

$$LH^{0*}(\mathcal{R}, \Lambda_9) = \mathbb{R},$$

$$\begin{aligned} LH^{1*}(\mathcal{R}, \Lambda_9) &= \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3, \\ LH^{3*}(\mathcal{R}, \Lambda_9) &= \oplus_{r\in\mathbb{N}} \mathbb{R} z^r \partial_{123}, \end{aligned}$$

and

$$LH^{2*}(\mathcal{R}, \Lambda_9) = \oplus_{r\in\mathbb{N}} H^2_r,$$

where

$$\begin{aligned} &H^2_0 = \mathbb{R}\partial_{23}, \quad H^2_1 = \mathbb{R}C^0_1, \quad H^2_3 = \mathbb{R}C^2_1, \\ &H^2_2 = \mathbb{R}x^2\partial_{23} + \mathbb{R}xz(\partial_{23} - 2^{-1}\partial_{31}) + \mathbb{R}(xz\partial_{12} - z^2\partial_{23}) \\ &\qquad +\mathbb{R}(yz\partial_{12} + (-27a^2x^2 - 9axz + 3ay^2 - z^2)\partial_{31}), \\ &H^2_{r+1} = \mathbb{R}C^r_1 + \mathbb{R}C^r_2, \quad r \geq 3, \end{aligned}$$

with

$$\begin{aligned} C^r_1 \quad = \quad & -a(xz^r + ry^2z^{r-1})\partial_{12} + (9a^2xy^r + a(3r-1)(r+1)^{-1}z^{r+1})\partial_{23} \\ & +ayz^r\partial_{31} \end{aligned}$$

and

$$\begin{aligned} C^r_2 \quad = \quad & (-a(r-2)y^4z^{r-3} + y^2z^{r-1})\partial_{12} \\ & +(9a^2xy^2z^{r-2} - 9ar^{-1}xz^r + 3a(r-3)(r-1)^{-1}y^2z^{r-1} - 3(r-1)r^{-1}(r+1)^{-1}z^{r+1})\partial_{23} \\ & +(6a(r-1)^{-1}xyz^{r-1} - ay^3z^{r-2} - r^{-1}yz^r)\partial_{31}, \end{aligned}$$

where the Y_i are those defined in Theorem 2 (and where the terms that contain a power of x, y, or z with a negative exponent are ignored).

7 Cohomological phenomena

Let us outline the most important cohomological phenomena.

Consider a SRMI Poisson structure $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$.

It is easily checked that the curl vector field of Λ, see Section 2, is given by $K(\Lambda) = \sum_i \delta_i Y_i$, $\delta_i = \operatorname{div} X_i$, $X_i = \sum_j \alpha^{ij} Y_j$. Consequently, K-exactness is (in $\mathbb{R}^n$, $n \geq 3$) equivalent with divergence-freeness. Note now that the 0–cohomology space $LH^{0*}(\mathcal{R}, \Lambda)$ of Λ, or space $\mathrm{Cas}(\Lambda)$ of Casimirs of Λ, coincides with the kernel $\ker \vec{X}$, see Equation 5. Hence, in view of Proposition 1, for a K-exact tensor, $D^p = (\det \ell)^p$ is a joint eigenvector of the X_i with eigenvalues $p\, \delta_i = 0$, i.e. $D^p \in \ker \vec{X}$. It follows that, for K-exact SRMI Poisson tensors,

$$\oplus_{p \in \mathbb{N}} \mathbb{R} D^p \subset LH^{0*}(\mathcal{R}, \Lambda) = \mathrm{Cas}(\Lambda).$$

As for the 1–cohomology space $LH^{1*}(\mathcal{R}, \Lambda)$, let us first remark that the stabilizer $\mathfrak{g}_\Lambda$, viewed as a Lie subalgebra of linear vector fields $\mathcal{X}^1_0(\mathbb{R}^n)$, is made up by 1-cocycles (by definition) that do not bound (degree argument), i.e.

$$\mathfrak{g}_\Lambda \subset LH^{1*}(\mathcal{R}, \Lambda).$$

Moreover, as LP-cohomology is an associative graded commutative algebra, the classes of the cocycles in

$$\mathrm{Cas}(\Lambda) \otimes \wedge^p \mathfrak{g}_\Lambda,$$

$0 \leq p \leq n$, are "preferential" LP-cohomology classes. Such classes massively appear in the LP-cohomology of SRMI tensors of the DHC, see [MP06], and of twisted SRMI tensors, see [AP07].

However, two other types of classes systematically appear in LP-cohomology.

1. The classes of type I originate from $\mathcal{P}$-cohomology. In fact, roughly spoken, the locus $\ker \vec{X}^{\mathbb{C}\,(1)}_\delta \oplus \ker \vec{X}^{\mathbb{C}\,(2)}_\delta \oplus \ldots \oplus \ker \vec{X}^{\mathbb{C}\,(s)}_\delta$ of the Koszul cohomology associated with the considered Poisson cohomology generates in some cases nonbounding cocycles in $\mathcal{R}$-cohomology. For instance, for structure Λ_7, the rational functions $D'^{\frac{\gamma}{2}} z^{-1}$, $D' = x^2 + y^2$, $\gamma \in 2\mathbb{N}^*$, induce the classes $D'^{\frac{\gamma}{2}} z^{-1} Y_3$, $Y_3 = z\partial_3$, in space $LH^{1*}(\mathcal{R}, \Lambda_7)$.

2. The classes of type II are due to $\mathcal{S}$-cohomology. Indeed, let $\mathfrak{s}$ be a cochain in space $\mathcal{S}$, which is supplementary to $\mathcal{R}$ in $\mathcal{P}$. It happens that $\partial_\Lambda \mathfrak{s} \in \mathcal{R}$. Then, $\partial_\Lambda \mathfrak{s}$—a coboundary of a cochain from the outside of $\mathcal{R}$—is typically a nonbounding cocycle in $\mathcal{R}$.

We refer to these two types of cohomology classes as "singular classes", since some of their coefficients are polynomials on the singular locus of the considered Poisson tensor.

Let us finally briefly comment on the impact of LP- and K-exactness on the structure of LP-cohomology. If tensor Λ, or part of this tensor, is LP-exact, see Section 2, some elements of space $\wedge^2 \mathfrak{g}_\Lambda$ may be bounding cocycles. For instance, part Y_{12} of structure Λ_3 of the DHC is LP-exact and disappears in the second cohomology space, see Theorem 8. Hence, LP-exactness impoverishes LP-cohomology. In view of the above remark on Casimir functions and the observations made in earlier works, we know that K-exactness significantly enriches the cohomology. Therefore, richness of LP-cohomology depends in some sense on the distance of the Poisson tensor to LP- and K-exactness.

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